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Higher-Order Corrections in Effective Theory of Deformed Nuclei

Jialin Zhang
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To the Graduate Council:

I am submitting herewith a thesis written by Jialin Zhang entitled "Higher-Order Corrections in Effective Theory of Deformed Nuclei." I have examined the final electronic copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Physics.

Thomas Papenbrock, Major Professor

We have read this thesis and recommend its acceptance:

Yuri A. Kamyshkov, Marianne Breinig

Accepted for the Council:
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Vice Provost and Dean of the Graduate School

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Higher-Order Corrections in Effective Theory of Deformed Nuclei

A Thesis Presented for

The Master of Science

Degree

The University of Tennessee, Knoxville

Jialin Zhang

August 2012

To My Beloved Parents

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Abstract

The low-energy excitation bands of open-shell heavy nuclei have been accounted for by collective motion of the constituting nucleons. Macroscopically, heavy nuclei can be looked upon as deformed rotors undergoing surface vibration and rotation. Traditionally, deformed nuclei are described within the Bohr-Mottelson geometric model or the interacting boson model. An effective theory that exploits spontaneous symmetry breaking has recently been developed for axially deformed nuclei. It describes the rotational and vibrational degrees of freedom in terms of Nambu-Goldstone bosons and quadrupole phonons respectively, with a power counting based on their different energy scales. A systematic way to construct the rotationally invariant Lagrangian under axial symmetry was established at next-to-leading order.

The purpose of this thesis is to extend the effective theory for deformed nuclei up to next-to-next-to-leading order. Higher-order corrections to Nambu-Goldstone modes and rotation-vibration coupling for both even-even nuclei and odd-mass nuclei are studied. For pure rotation, higher-order Nambu-Goldstone modes prove to only perturb the energy spectrum by the corresponding powers of the leading-order eigenenergy. As expected, the next-to-next-to-leading-order calculation of Nambu-Goldstone modes exhibits a higher accuracy than next-to-leading order after fitting to experimental level schemes. When vibration is coupled to rotation, the next-to-leading-order Hamiltonian correctly yields the rotational-vibrational spectrum of deformed nuclei. In the derivation of rotation-vibration coupling Hamiltonian in even-even nuclei, a perturbative method (Fukuda's inversion method) for the Legendre

transformation is employed. The effect of next-to-next-to-leading-order rotation-vibration coupling yields a correction of the moment of inertia that depends on the vibrational band head. Furthermore, for odd-mass nuclei with finite ground-state spins, a gauge invariant Hamiltonian is obtained. While the rotational band encompasses a combined contribution from intrinsic spin and azimuthal angular momentum, the vibrational spectrum exhibits a feature of Landau levels with a high density of states, stemming from the velocity-dependent Lagrangian.

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Nomenclature

α, β	Euler angles specifying the orientation of a deformed nucleus–parameter of Nambu-Goldstone modes
ΔL_{NNLO}	Rotation-vibration coupling Lagrangian at NNLO
$\dot{a}$	First-order time derivatives of a general variable a
γ	Angular component of ψ_2 in polar coordinates
$\hat{J}_z$	Operator of the component of angular momentum along z-axis (symmetry axis of a deformed nucleus)
$\hat{l}_2$	Azimuthal angular momentum operator of an axially symmetry harmonic oscillator
ω'	Redefinition of $\widetilde{B}$ which specifies the interaction strength between a deformed nucleus and a magnetic field
Ω	Energy scale of high-energetic excitations associated with quadrupole vibrations, pairing, and other single-particle degrees of freedom
ω_0, ω_2	Angular frequencies of the axially symmetric harmonic-oscillator potential
ϕ	Parameterized quadrupole field excluding Nambu-Goldstone modes, with three degrees of freedom

ϕ_i	i-th component of the quadrupole field ϕ
ψ	Unparameterized quadrupole field with five degrees of freedom
ψ_i	i-th component of the quadrupole field ψ
LO	leading-order or leading order
NLO	next-to-leading-order or next-to-leading order
NNLO	next-to-next-to-leading-order or next-to-next-to-leading order
φ_0	$\phi_0 - v$
φ_2	Amplitude of ψ_2 in polar coordinates
ξ	Energy scale of low-energetic excitations associated with Nambu-Goldstone modes
A, B, $\widetilde{B}$	Power counting constants in front of time-odd terms in Lagrangian of odd-mass nuclei
C_0, C_2, C_4	Power counting constants in Nambu-Goldstone modes to be determined by fitting to data
D_0, D_1, D_2	Power counting constants in rotation-vibration coupling to be determined by fitting to data
D_t	Covariant derivative operator
$E_\pm$	$E_x \mp iE_y$
$E_{\rm LO}, E_{\rm NLO}, E_{\rm NNLO}$	Eigen energies of the LO, NLO and NNLO Hamiltonians
E_x, E_y, E_z	Components of the Nambu-Goldstone field
F_0, F_1, F_2	Power counting constants in rotation-vibration coupling to be determined by fitting to data

$H_{\mathrm{LO}}, H_{\mathrm{NLO}}, H_{\mathrm{NNLO}}$	LO, NLO and NNLO Hamiltonians
J_z	The quantum number of $\hat{J}_z$
K	Projection of total angular momentum on symmetry axis / Ground-state spin quantum number
l	Total angular momentum quantum number
l_2	Quantum number of $\hat{l}_2$
$L_{\mathrm{LO}}, L_{\mathrm{NLO}}, L_{\mathrm{NNLO}}$	LO, NLO and NNLO Lagrangians
L_{WZ}	Wess-Zumino term in a rotationally invariant Lagrangian
n_0, n_2	Primary quantum numbers of an axially symmetry harmonic oscillator
p_a	Conjugate momentum of the general variable a
Q	Magnitude of total angular momentum of a deformed nucleus
q	Ground-state spin quantum number
R, S	Power counting constants that redefine D_0, D_2, F_0, F_2, ω_0 and ω_2
v	Vacuum expectation value of zero component of quadrupole field ϕ
V_{LO}	Potential of a 3-dimensional harmonic oscillator with axial symmetry
x	x-component of ψ_2 in x-y coordinates
y	y-component of ψ_2 in x-y coordinates

Chapter 1

Introduction

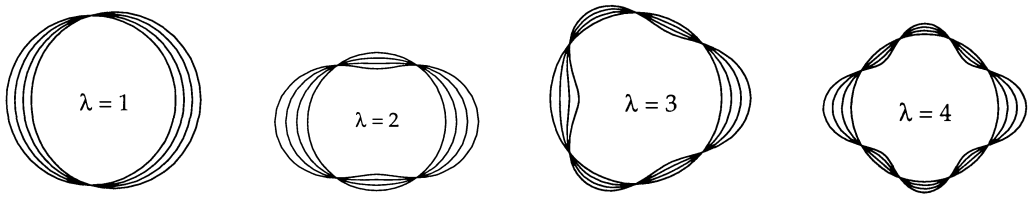
The prosperity of the modern world has been dramatically propelled by the exploitation of atomic nuclei: nuclear power, nuclear medicine, environmental tracers, just to name a few. The study of nuclear structure and dynamics is a subject of enduring and significant interest, for the reason of both its immensurable application potential and its profound connections with other branches of modern physics. The fact that atomic nuclei are a collection of species ranging widely in the way of proton-neutron composition and manifesting a vast variety of phenomena, predicates that a comprehensive understanding of nuclear structure must rely on a clear identification of distinct aspects of the nuclear features, as well as an appreciation of different models. The two remarkable models that provide a foundation for our understanding are the shell model [2, 3] and the collective model [4, 5], upon which most following development of nuclear theory is based.

The shell model, focusing on the individual motions of the nucleons, assumes a spherical potential for the nucleon-nucleon interaction and aims to solve a many-body eigenvalue problem in principle. While the residual interactions arising from the departure from closed shell are often considered weak enough to be treated as perturbations, the goal of reaching a self-consistent solution to the many-body Schrödinger equation can be generally achieved by means of the Hartree-Fock

approach within mean-field theory. Recent progresses in the shell model have managed to overcome many earlier drawbacks it suffered. Novel numerical algorithms such as Lanczos construction, Monte Carlo methods, have been greatly developed to accelerate the matrix diagonalization. Large quadrupole moments and strong transitions between low-lying excitation states of a number of heavy nuclei in the mid-shell region have been obtained by the shell-model calculation [6]. The rotational motion has been successfully brought into the context of the spherical shell model by SU(3) symmetry [7, 8].

While the shell model is grounded on an independent-particle point of view, its limitations are strongly recognized by physicists, particularly in handling medium and heavy nuclei. The dimensionality of the matrices in Slater determinantal spaces is often so huge that they cannot be diagonalized at present. For example, in $^{154}_{62}$Sm up to around 10^{14} states can be constructed for 0^+ level in the major shell [9]. Although the overwhelming capability of computational power nowadays has to a large extent enabled such calculations for rare-earth nuclei, the process is usually intractable and the final results are hard to interpret. Moreover, the shell model, adopting a spherical configuration from the beginning, neglects an important fact that many nuclei are intrinsically deformed at their equilibrium state.

These problems are well addressed by the pioneering work of Bohr [4], who was later joint by Mottelson [5] in 1950s, with the proposal of a collective description of the atomic nuclei. They depict the nucleons to be moving coherently in the atomic nucleus which exhibit collective behavior, whose motivation can be traced back to the very first model of nuclear theory - the liquid drop model. The collective motion can be decomposed into many modes, with the central ones being the surface vibration and rotation. Figure 1.1 shows the most basic modes of surface vibration, among which the quadrupole mode is the one most commonly detected. Figure 1.2 is a rotation scheme of a deformed nucleus, which essentially is analogous to a rigid rotor. One of the glorious triumphs of the collective model is that it successfully explains the low-lying excitation bands of heavy nuclei associated with rotation, which is of



(a) dipole **(b)** quadrupole **(c)** octupole **(d)** hexadecupole

Figure 1.1: Basic modes of surface vibration

the form

$$E_{l,K} = \frac{\hbar^2}{2\mathcal{I}}\big(l(l+1) - K(K+1)\big)\ . \tag{1.1}$$

Here, $\mathcal{I}$ is the moment of inertia and l is the total angular momentum quantum number. K is the quantum number of the projection of total angular momentum along the symmetry axis of the deformed nucleus, which characterizes the excitations of intrinsic angular momentum (i.e. spin). Since the total angular momentum is the superposition of the orbital part and the spin, l can only take on the values $l \geq K$.

If one looks at the even-even nuclei in the chart of nuclides (Figure 1.3), the energy ratios E_{4^+}/E_{2^+} in the rare-earth and actinide regions are mostly close to 10/3, which signifies a rotational band of formula Eq.1.1 and implies the deformability of the rare-earth nuclei. Also note that open-shell nuclei favor a deformed shape in their ground state, while spherical nuclei are typically found around closed shells.

A typical level scheme that demonstrates the deexcitation of a deformed nucleus (^{162}Dy here, taken from Ref. [1]) in the collective excitation region is shown in

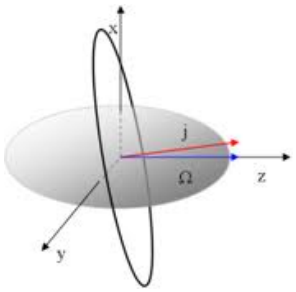


Figure 1.2: Collective rotation of a deformed nucleus

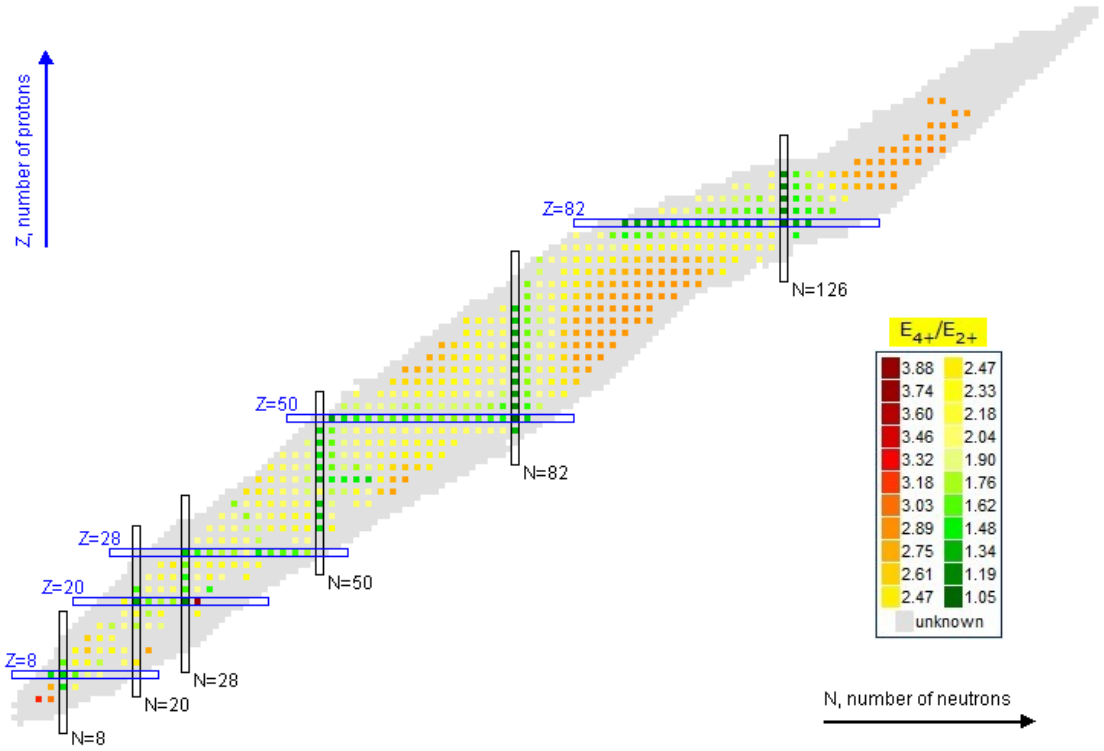


Figure 1.3: Chart of nuclides for even-even nuclei (from National Nuclear Data Center, URL: http://www.nndc.bnl.gov/chart/)

Figure 1.4. There are three rotational bands in this level scheme of ^{162}Dy: the ground-state band with band-head $J^\pi = 0^+$, the first excited β-band with band-head $J^\pi = 0^+$ and the first excited γ-band with band-head $J^\pi = 2^+$. Experimentally, a complete level scheme of low-lying excitations of a deformed nucleus is established through spectroscopic measurements following thermal neutron capture on the odd-mass isotope of the even-even deformed nucleus. For instance, levels in ^{162}Dy can be populated by a non-selective, high-flux neutron beam illuminating an enriched target of $^{161}Dy_2O_3$. Two principal reactions exploited to determine the states information are the ^{161}Dy(n,γ)^{162}Dy yielding a γ-ray spectrum, and the ^{161}Dy(n,γ)(γ,e$^-$)^{162}Dy yielding a conversion electron spectrum. The γ spectrum is measured by curved-crystal spectrometer on the principle of Bragg diffraction, and the conversion electron spectrum is measured by β spectrometer.

J^π values of the states can be derived from the transition multipolarities, which

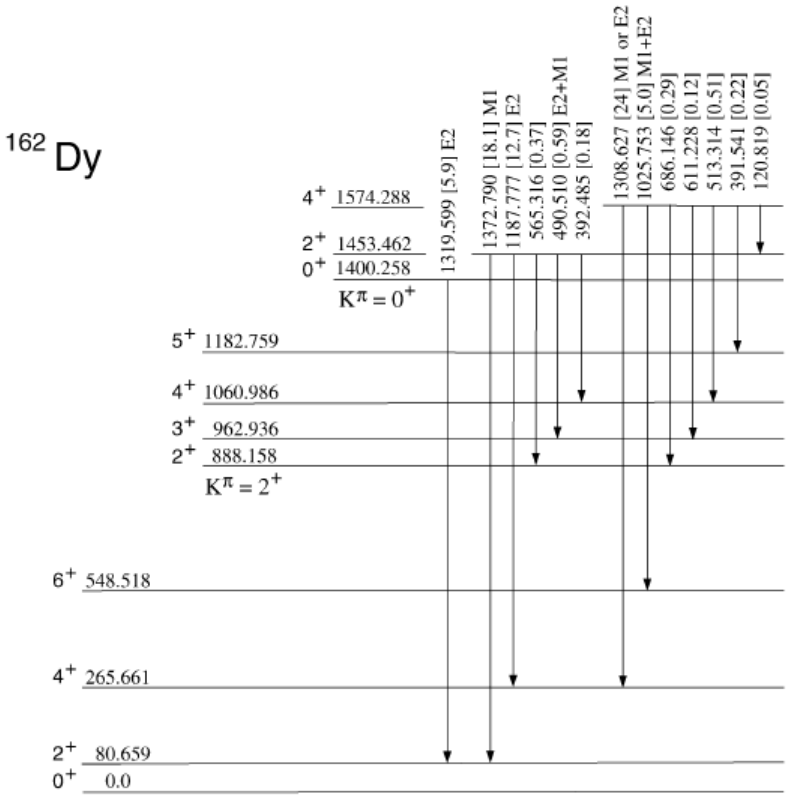


Figure 1.4: Partial level scheme of ^{162}Dy showing the depopulation of the first excited $K^\pi = 0^+$ band at 1400.258 keV, with γ-ray transition energies, intensities and multipolarities as indicated. (taken from Ref. [1])

are either directly observed in γ transitions or deduced from the measured internal conversion coefficients on the basis of theoretical values. The completeness of the low-spin levels is guaranteed by average resonance capture studies. The distinction between levels with same spin but in different bands can be made with the theoretical argument that intra-band transition strengths are strong and interband transitions have relatively weak strength. This results from the calculations of intrinsic matrices linking low-lying collective bands. While positive-parity bands are described in terms of quadrupole vibrations, bands with negative parity can be interpreted by octupole vibrations. For a detailed description of experiments that measure complete spectra of deformed nuclei, reader can refer to Ref. [1, 10].

Implicit in our discussion is that in many cases it is adequate, at least at the

leading order, to approximate the deformed nuclei as maintaining the axial symmetry, i.e. variations along any direction in the plane perpendicular to the symmetry axis are equal. The quadrupole deformation of the surface of axially deformed nuclei is described by five degrees of freedom. The original Bohr-Mottelson model uses two shape parameters measuring the extent of the deformation and three Euler angles measuring the relative position between the co-rotating intrinsic reference frame and laboratory frame. Extension to the Bohr-Mottelson model has been made to deal with triaxially deformed nuclei within the asymmetric rotor model [11]. While the important role of symmetry has been realized since the beginning of the collective model (See Bohr's Nobel lecture [12]), it is not until recently an effective theory based on spontaneous breaking of rotational symmetry is invented for deformed nuclei, which provides a model-independent view of the collective excitations [13].

Effective field theories (EFT) have been routinely used in physics systems that display a separation of energy scales, for its impressive efficiency in computing physical observables. They have gained increasing popularity and success in recent years; in low-energy nuclear physics, for example, the description of few-nucleon systems within chiral EFT [14, 15, 16], the application of EFT to halo nuclei [17, 18] and EFT for dilute fermi systems [19, 20, 21]. EFT are first developed by Weinberg to study nucleon-nucleon and nucleon-pion interactions based on spontaneous symmetry breaking of chiral symmetry [22, 23]. In our case of deformed nuclei, due to the finite size of atomic nucleus, the descriptive fields in the spontaneous breaking of rotational symmetry are only functions of time (and not of space), so that one simply deals with quantum mechanics rather than field theory [13].

Within the effective theory, one starts by identifying the relevant pattern of the symmetry breaking (in our case: the rotational symmetry for deformed nuclei) and distinguishing the degrees of freedom at different energy scales, then proceeds with the hunt for building blocks of an invariant Lagrangian under the corresponding symmetry, through applying the principle of spontaneous symmetry breaking. A power counting has to be established to allow for the separation of scale and a

systematic extension to higher order. For deformed even-even nuclei in the rare-earth region, the vibrational degrees of freedom are described in terms of the quadrupole phonons and the rotational degrees of freedom are described by Nambu-Goldstone bosons. Validity of the power counting to separate vibration and rotation is justifiable for the fact that the first rotational excited state $J^\pi = 2^+$ states is at several tens of keV of excitation energy, whereas the first vibrational 2^+ state is at about 1 MeV, i.e. approximately larger by two orders of magnitude. Both low-lying vibrational and rotational spectra at the leading order, are immediately obtained upon the quantization of the Hamiltonian [13].

There are many phenomenological models that seek a description of collective phenomena in atomic nuclei, such as general collective model [24], variable moment of inertia model [25] and the interacting boson model [26, 27]. However, they are difficult to be systematically extended and it is difficult to reliably gauge their accuracy. Furthermore, attempts to generalize the collective model and to keep it computationally tractable turn out to be non-trivial, primarily because of the linear realization of rotational symmetry. The advantage of employing a non-linear realization of the rotational symmetry group (namely, the choice of different coordinate systems with Nambu-Goldstone modes and quadrupole fields), has been demonstrated in constructing the Lagrangian and computing the conserved quantities [13].

The purpose of this thesis is to extend to higher orders within the effective theory for deformed nuclei, and to go beyond the phenomenological models. The leading-order (LO) and next-to-leading-order (NLO) calculations have been completed in Ref. [13]. At this order, one finds an exact agreement with the phenomenological models. For single rotational bands described by pure Nambu-Goldstone modes, we will see that not only the next-to-next-to-leading-order (NNLO) corrections indeed give rise to higher accuracy once fitted to level schemes, but also the orders of magnitude of the fitting coefficients agree with the estimates of power counting.

One of the main results of this thesis is that in even-even deformed nuclei

the rotation-vibration coupling at next-to-next-to-leading order yields corrections to the rotational-vibrational spectra that can modify the moment of inertia either increasingly or decreasingly. The phenomenological geometric models predict a slightly reduced moment of inertia at higher vibrational excitation states, in contradiction to the experimental observations. It is more intuitive and reasonable to have a larger moment of inertia at higher excited states due to the centrifugal stretching of the nuclei originating from higher angular velocity. In the EFT, odd-mass nuclei are studied on an equal footing with even-even nuclei, but with additional time-odd terms entering the Lagrangian. These terms lead to a suppressing effect to the energy spectra of odd nuclei, which characteristically is the same with Landau levels in the presence of a constant magnetic field. Technically, we employ a perturbation method – Fukuda's inversion method – to perform the Legendre transformation when deriving the rotationally invariant Hamiltonian. It is to overcome the complexity of the rotation-vibration coupling terms that admix different degrees of freedom. The correctness and completeness of the higher-order corrections are assured by the power counting throughout the derivation.

It should be pointed out that the collective property and single-particle property of deformed nuclei are in nature deeply related. Efforts to unify the collective model with the shell model have never stopped [28]. The Nilsson model, as the first one doing so and enjoying considerable success, suggests a deformed mean field adapted to the shell structure for deformed nuclei [29]. The concept of spontaneous symmetry breaking has also been applied to rotating mean field of nuclei under the cranking model, which is specialized for high-spin phenomena [30, 31]. Intensive researches have also been conducted on the calculation of parameters of collective Hamiltonian by microscopic approaches [32, 33, 34]. The aim of the effective theory at higher orders is not to pursue a unified approach that incorporates microscopic mechanism into the collectivity, but rather to present an alternative and systematic formalism. My contribution to the effective theory of deformed nuclei in current study primarily lies in the modification to moment of inertia at the excited rotational bands, which

can take on either positive or negative values.

The thesis is organized as follows. Chapter 2 studies the physics of pure Nambu-Goldstone modes at higher orders, with a focus on individual rotational bands. In Chapter 3, we couple the quadrupole vibration to rotation for even-even nuclei, and study the higher-order corrections to the energy spectra resulting from the rotation-vibration coupling. Odd-mass nuclei are investigated in Chapter 4 (which is a verification of Papenbrock's unpublished work). The results in each chapter are illustrated and supported by comparison to experimental low-excitation spectra of representative nuclei. A summarization of major findings and a brief outlook are given in Chapter 5. Current progress pertaining to Nambu-Goldstone modes with second-order time derivatives is presented in detail in the Appendix.

Chapter 2

Individual Rotational Bands

A system with spherical symmetry is rotationally invariant. A deformation of such a system introduces a difference in orientation, and thus breaks the rotational symmetry. For deformed nuclei, the rotational degrees of freedom arising from the collective mode of excitations, naturally lead one to recall the idea of spontaneously symmetry breaking. In a strict sense, spontaneous symmetry breaking only happens in idealized systems that possess an infinitely large number of degrees of freedom [35]. Ferromagnets and superconductors are examples in this approximation, with the former spontaneously breaking the rotational symmetry and the latter breaking the local gauge symmetry.

An atomic nucleus has far less degrees of freedom compared to systems like ferromagnets or superconductors. Nor could the discrete excitation states in the rotational band mix together under an arbitrarily small perturbation. In other words, an atomic nucleus at ground state is in general stable, thus not subject to deformation against an infinitesimal perturbation. Nevertheless, the rationality to exert spontaneous symmetry breaking on deformed nuclei can be justified by the separation of energy scales between different degrees of freedom in the collective excitations. For instance, the first rotational excited states of both ^{162}Dy and ^{172}Yb are of about 80 keV of excitation energy, while their band heads of vibrational

excitations are approximately at 1 MeV. This is generally the case for nuclei in the rare-earth region. An energy that is not infinitesimally small, but much smaller than any other excitation modes, will still be sufficient to disturb the rotational band and admix the states with different angular momenta, thus inducing a spontaneous breaking of the rotational symmetry.

Both theoretically and experimentally, of particular interest is the nuclei deformation that preserves the axial symmetry, for its simplicity and commonality. The ground state of such axially deformed nuclei is not invariant under a general operation of the full rotation group $\mathcal{G} = SO(3)$. However, it is invariant with respect to operations of the subgroup $\mathcal{H} = SO(2)$, which is around the symmetry axis. Procedures to systematically construct Lagrangian that is invariant under the subgroup $\mathcal{H}$ on the basis of spontaneous symmetry breaking has been fully demonstrated [13, 35]. Interested readers can refer to Ref. [36, 37, 38] for a description about constructing low-energy Lagrangians based on spontaneous symmetry breaking. A generalized approach for non-relativistic effective Lagrangians can be found in [39].

More specifically, it is the coset $SO(3)/SO(2)$ that mathematically describes the rotational degrees of freedom. The appearance of Nambu-Goldstone bosons in the spontaneous breakdown of a continuous symmetry naturally provides us with a means to parameterize the coset. In other words, the Nambu-Goldstone modes correspond to the collective rotation of deformed nuclei.

The building blocks of an invariant Lagrangian under the subgroup $\mathcal{H}$ are E_+, E_-, ϕ, and $D_t\phi$ [13], where

$$E_\pm = E_x \mp iE_y \ , \tag{2.1}$$

$$D_t \equiv \partial_t - iE_z J_z \ . \tag{2.2}$$

$$E_x = \dot{\alpha} \sin\beta \ ,$$
$$E_y = -\dot{\beta} \ ,$$
$$E_z = -\dot{\alpha} \cos\beta \ . \tag{2.3}$$

Here $\alpha(t)$ and $\beta(t)$ parameterize the coset $SO(3)/SO(2)$ and are Euler angles that determine the orientation of the axially symmetric deformed nucleus in the laboratory frame [40]. D_t is the covariant derivative operator, and J_z is the operator or quantum number with regard to the component of angular momentum along the symmetry axis. ϕ is a field that describes physics at energy scales higher than the rotation. The vibration of even-even nuclei is appropriately described by the quadrupole field. The transformation properties of each building block under a general rotation have also been derived [13]. The transformation properties lay the foundation for us to construct rotationally invariant Lagrangians within the framework of our effective theory. Using the well-established power counting based on dimension analysis and physical arguments, one can always compose rotationally invariant terms and organize them into the Lagrangian up to the desired order.

In this chapter, we focus on pure Nambu-Goldstone modes related to the individual rotational band of deformed nuclei. The corrections to the spectra of the rotational excitations are calculated up to next-to-next-to-leading order. In the presence of vibration, one must introduce quadrupole phonons described by a quadrupole field ψ. The field ϕ in the "building blocks" is a transformation of the original field ψ, which fulfills the reality relation and discards the redundant degrees of freedom that have already been included by Nambu-Goldstone modes [13]. We will see this in next Chapter.

2.1 Spectrum in next-to-next-to-leading order

For even-even nuclei whose ground state is invariant under time reversal, the Lagrangian of Nambu-Goldstone modes up to next-to-leading order has been written

out [13]

$$L_{\rm LO} = \frac{C_0}{2} E_+ E_- = \frac{C_0}{2}(\dot{\alpha}^2 \sin^2\beta + \dot{\beta}^2) \ , \tag{2.4}$$

$$L_{\rm NLO} = L_{\rm LO} + \frac{C_2}{4}(E_+ E_-)^2 = L_{\rm LO} + \frac{C_2}{4}(\dot{\alpha}^2 \sin^2\beta + \dot{\beta}^2)^2 \ . \tag{2.5}$$

For odd-mass and odd-odd nuclei, an additional Wess-Zumino term comes into the LO Lagrangian, so as to satisfy the condition that the ground state is no longer invariant under time reversal due to the finite spin. It turns out in our effective theory that the Wess-Zumino term can be expressed by [13]

$$L_{\rm WZ} \equiv q E_z = -q\dot{\alpha}\cos\beta \ , \tag{2.6}$$

so that at leading order, the Lagrangian becomes

$$L_{\rm LO} = \frac{C_0}{2}(\dot{\alpha}^2 \sin^2\beta + \dot{\beta}^2) - q\dot{\alpha}\cos\beta \ , \tag{2.7}$$

and the NLO Lagrangian keeps the same expression with Eq.2.5.

The LO energy eigenvalues and eigenstates of both even-even and odd nuclei have been calculated. At the next-to-leading order, the Hamiltonians of Nambu-Goldstone modes in both cases end up with the same form if written in terms of the LO Hamiltonian. The NLO correction to the rotational spectrum in pure Nambu-Goldstone modes is simply two powers of the LO eigenenergy [13], which is exactly consistent with the results of Bohr-Mottelson model [5] and other phenomenological models. Now we follow the track down to calculate the NNLO correction in Nambu-Goldstone modes. Assuming the LO energy scale of Nambu-Goldstone modes is of order ξ, i.e. $E_{\rm LO} \sim \xi$, the dimension analysis at the leading order yields [13]

$$\begin{aligned} C_0 &\sim \xi^{-1} \ , \\ \dot{\alpha} \sim \dot{\beta} \sim E_{x,y,z} &\sim \xi \quad . \end{aligned} \tag{2.8}$$

A further assumption in our effective theory is that the low energy scale of rotation is well separated from high energy scales of other degrees of freedom (e.g. vibrational, pairing, nucleonic degrees of freedom). The low energy is associated with Nambu-Goldstone modes $\sim \xi$, and we denote the high energy scale as Ω, where $\Omega \gg \xi$. In rare-earth nuclei, $\Omega \sim 1$ MeV and rotational excitations ξ are about 80 keV. The entire spectrum can be expanded like a Taylor series so that the ratio between the higher-order correction and the LO energy must be the powers of ξ/Ω. To be more concrete, the interactions between Nambu-Goldstone modes and high-energy excitations, and among Nambu-Goldstone bosons themselves, essentially give rise to higher-order corrections to the LO rotational bands, which can be factorized by the powers of ξ/Ω. It is required that $\xi/\Omega \ll 1$ to fulfill the assumption of the effective theory. Otherwise, excitations of different degrees of freedom tend to intertwine and our effective theory will break down. On the other hand, it is satisfactory that in the limit $\Omega \to \infty$, all of the higher-order corrections will vanish, which leaves us only the exact solutions to the eigen equations of leading order.

We look at the next-to-next-to-leading-order (NNLO) correction. The Lagrangian becomes

$$L_{\rm NNLO} = L_{\rm NLO} + \frac{C_4}{6}(E_+E_-)^3 = L_{\rm NLO} + \frac{C_4}{6}(\dot{\alpha}^2\sin^2\beta + \dot{\beta}^2)^3 \,. \tag{2.9}$$

Here C_4 is the low-energy coefficient that is related to the omitted physics at the breakdown scale ($\sim \Omega$) and needs to be determined by fitting to data. However, the dimension analysis and power counting will help us estimate its size before resorting to the experimental data. According to the power counting, ratio of NNLO correction to LO energy must be

$$\frac{C_4(E_+E_-)^3}{C_0(E_+E_-)} = \frac{C_4}{C_0}(E_+E_-)^2 \sim \left(\frac{\xi}{\Omega}\right)^4 \,, \tag{2.10}$$

so that

$$\frac{C_4}{C_0} \sim \Omega^{-4} \,. \tag{2.11}$$

The NNLO correction is suppressed by a factor of four powers of ξ/Ω compared to the LO energy.

To obtain the Hamiltonian we perform the Legendre transformation. The conjugate momenta are

$$p_\alpha = \frac{\partial L_{\rm NNLO}}{\partial \dot{\alpha}} = \left(C_0 + C_2 \left(E_+ E_-\right) + C_4 \left(E_+ E_-\right)^2\right) \dot{\alpha} \sin^2\beta - q\cos\beta \,, \tag{2.12}$$

$$p_\beta = \frac{\partial L_{\rm NNLO}}{\partial \dot{\beta}} = \left(C_0 + C_2 \left(E_+ E_-\right) + C_4 \left(E_+ E_-\right)^2\right) \dot{\beta} \,. \tag{2.13}$$

This yields the Hamiltonian

$$\begin{aligned} H_{\rm NNLO} &= p_\alpha \dot{\alpha} + p_\beta \dot{\beta} - L_{\rm NNLO} \\ &= \frac{C_0}{2}\left(E_+ E_-\right) + \frac{3C_2}{4}\left(E_+ E_-\right)^2 + \frac{5C_4}{6}\left(E_+ E_-\right)^3 \end{aligned} \tag{2.14}$$

The LO Hamiltonian and eigenenergy is essentially a classical rotor [13]

$$\begin{aligned} H_{\rm LO} &= \frac{\left(p_\alpha + q\cos\beta\right)^2}{2C_0 \sin^2\beta} + \frac{p_\beta^2}{2C_0} \,, \\ E_{\rm LO} &= \frac{l(l+1) - q^2}{2C_0} \,. \end{aligned} \tag{2.15}$$

where q is the spin of the band head.

For simplicity, we further express it in terms of position and velocity, instead of

position and momentum

$$
\begin{aligned}
H_{\mathrm{LO}} &= \frac{(p_\alpha + q\cos\beta)^2}{2C_0\sin^2\beta} + \frac{p_\beta^2}{2C_0} \\
&= \frac{1}{2C_0}\left(C_0 + C_2\left(E_+E_-\right) + C_4\left(E_+E_-\right)^2\right)^2\left(\dot{\alpha}^2\sin^2\beta + \dot{\beta}^2\right) \\
&\approx \frac{1}{2C_0}\left(C_0^2 + 2C_0C_2(E_+E_-) + C_2^2(E_+E_-)^2 + 2C_0C_4(E_+E_-)^2\right)(E_+E_-) \\
&= \frac{C_0}{2}(E_+E_-) + C_2(E_+E_-)^2 + \left(\frac{C_2^2}{2C_0} + C_4\right)(E_+E_-)^3 \,. \qquad (2.16)
\end{aligned}
$$

In each step, the Hamiltonian is truncated to the NNLO precision which is what we need. Higher-order terms than NNLO can be neglected at all from a practical point of view. Likewise, up to NNLO the square and the cubic of Hamiltonian are

$$
H_{\mathrm{LO}}^2 = \frac{C_0^2}{4}(E_+E_-)^2 + C_0C_2(E_+E_-)^3 \,, \qquad (2.17)
$$

$$
H_{\mathrm{LO}}^3 = \frac{C_0^3}{8}(E_+E_-)^3 \,. \qquad (2.18)
$$

Finally we wish to express H_{NNLO} in terms of H_{LO}. The equation to be solved is quite straightforward, and

$$
H_{\mathrm{NNLO}} = H_{\mathrm{LO}} - \frac{C_2}{C_0^2}H_{\mathrm{LO}}^2 + 4\left(\frac{C_2^2}{C_0^4} - \frac{C_4}{3C_0^3}\right)H_{\mathrm{LO}}^3 \,. \qquad (2.19)
$$

The perturbation theory gives the spectrum of Nambu-Goldstone modes at once

$$
E_{\mathrm{NNLO}} = E_{\mathrm{LO}} - \frac{C_2}{C_0^2}E_{\mathrm{LO}}^2 + 4\left(\frac{C_2^2}{C_0^4} - \frac{C_4}{3C_0^3}\right)E_{\mathrm{LO}}^3 \,. \qquad (2.20)
$$

We see the second term is equivalent with what has been derived for the NLO correction [13]. The third term acts as a NNLO correction to the rotational band. If

we check the order of magnitude of the third term, with Eq.2.11 there are

$$\frac{C_2^2}{C_0^4} \sim \xi^2\Omega^{-4}\,, \tag{2.21}$$

$$\frac{C_4}{C_0^3} \sim \xi^2\Omega^{-4}\,, \tag{2.22}$$

thus,

$$\frac{C_2^2}{C_0^4} - \frac{C_4}{3C_0^3} \sim \xi^2\Omega^{-4}\,, \tag{2.23}$$

$$\left(\frac{C_2^2}{C_0^4} - \frac{C_4}{3C_0^3}\right) E_{\mathrm{LO}}^3 \sim \xi^5\Omega^{-4} = \xi\left(\frac{\xi}{\Omega}\right)^4\,. \tag{2.24}$$

Indeed, the order of magnitude of the new correction term is of next-to-next-to-leading order as we expected. The NNLO energy E_{NNLO} is a polynomial of degree of three of the LO energy. This is also in agreement with the calculations of all hitherto phenomenological models. Remember that we are simply using a semi-phenomenological approach where only the principles of spontaneous symmetry breaking are needed. Though the calculation here is done for general nuclei, it is exactly the same in the case of even-even nuclei, for which $q = 0$.

The unknown coefficients C_0, C_2 and C_4 are to be determined by fitting the spacing between the first several rotational states in an individual rotational band. The value obtained by fitting should always be compared with the power counting results, to ensure the validity of the higher-order corrections in our effective theory. This is the task of next section.

2.2 Comparison with level schemes

The main result about NNLO correction in Nambu-Goldstone modes is the rotational energy spectrum Eq.2.20. In this section, we aim to compare it with

realistic experimental data, and analyze it detailedly and make further physical arguments.

To ease the job, we first rewrite Eq.2.20 in a simpler form. Since the LO eigenenergy E_{LO} is [13]

$$E_{\mathrm{LO}} = \frac{1}{2C_0} l(l+1) \ . \tag{2.25}$$

Here, l is the total angular momentum quantum number.

E_{NNLO} can be expressed as

$$E_{\mathrm{NNLO}} = al(l+1) + b\left(l(l+1)\right)^2 + c\left(l(l+1)\right)^3 \ . \tag{2.26}$$

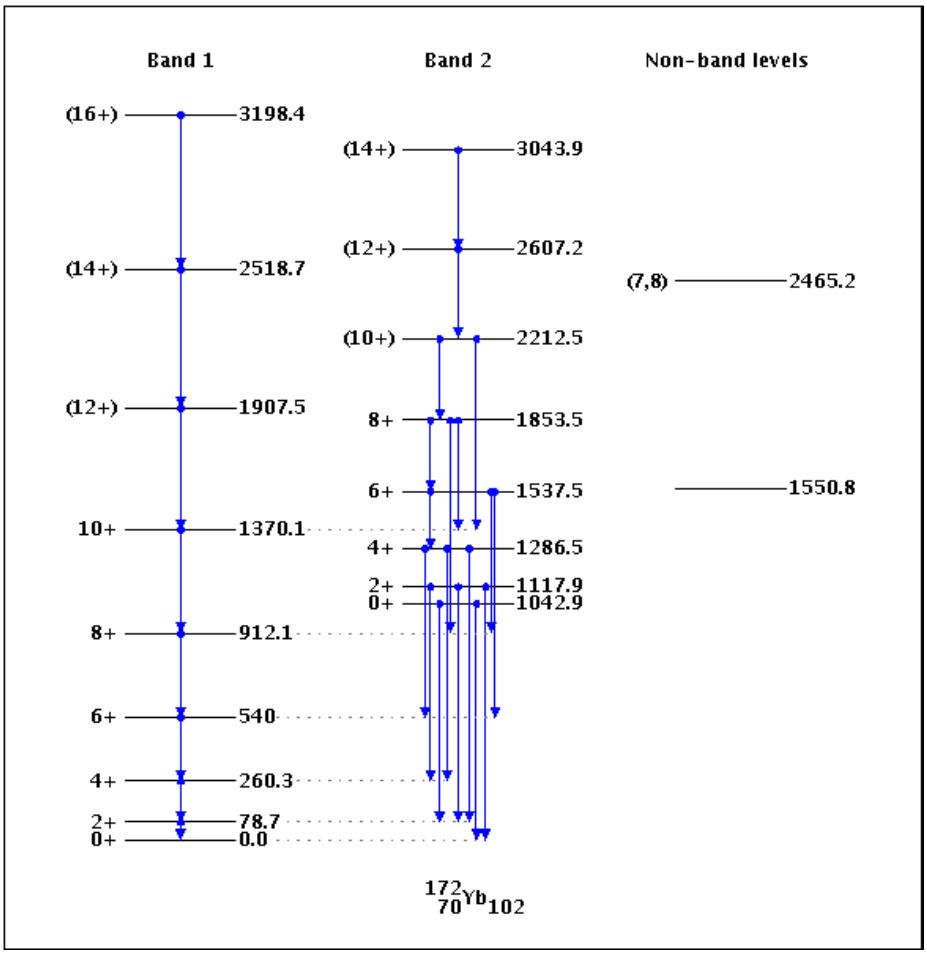
Parameters a, b and c are a redefinition of C_0, C_2 and C_4, and are more straightforward to use in fitting with level schemes of the rotational excitations. Obviously, their order of magnitude should be scaled as follows according to the scaling of C_0, C_2 and C_4,

$$\begin{aligned} a &\sim \xi \ , \\ b &\sim \xi \left(\frac{\xi}{\Omega}\right)^2 \ , \\ c &\sim \xi \left(\frac{\xi}{\Omega}\right)^4 \ . \end{aligned} \tag{2.27}$$

Below we will see that this is verified by all the results of the fitting.

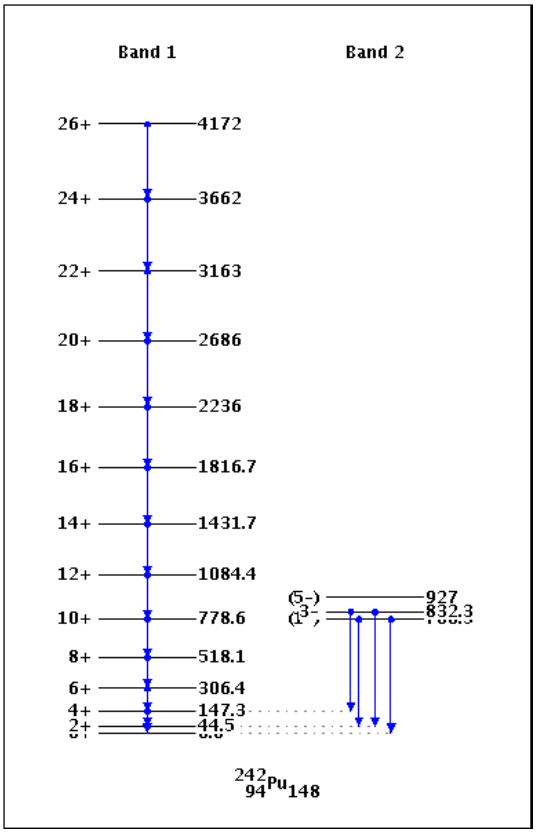
We choose ^{172}Yb, ^{242}Pu and ^{238}U, whose ground-state rotational bands are shown in Figure 2.1 (a), (b) and (c), respectively (Band 1 in each scheme). Each entire single band is fitted by a polynomial function of $l(l+1)$ in Origin8.6, with the intercept being imposed to be zero. The highest order of the polynomial function is either two or three, depending on whether we are fitting in next-to-leading order or next-to-next-to-leading order.

The results of fitting are shown in Table 2.1 and 2.2, with regard to NLO and NNLO respectively. As we can see, in both NLO and NNLO fitting, the coefficients a, b, and c are all of the right orders of magnitude derived by the power counting.

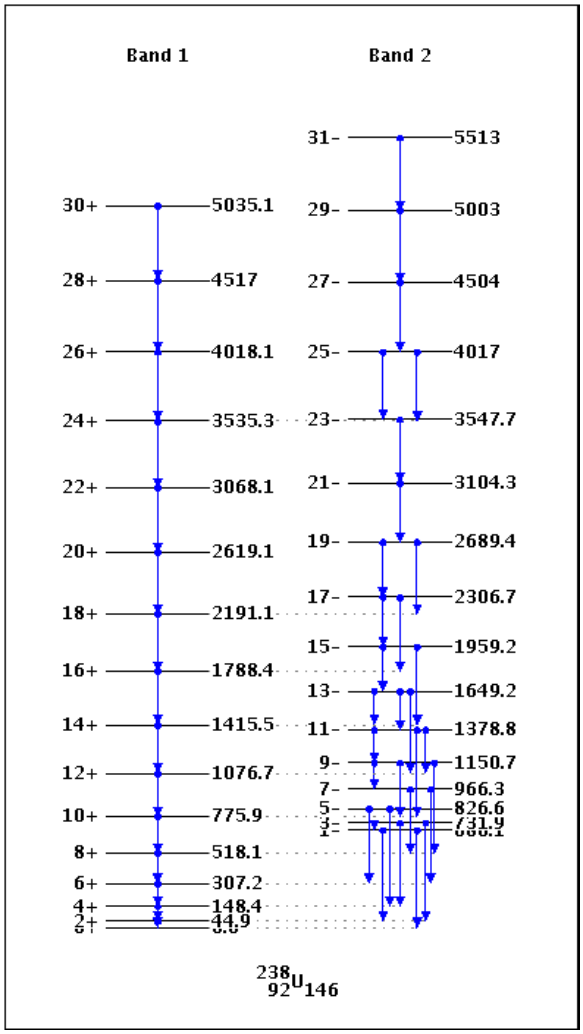


(a) ^{172}Yb

Figure 2.1: Level schemes of single rotational bands. Spin and parity as indicated, energies in keV. (Band 1 for fitting. From National Nuclear Data Center, URL: http://www.nndc.bnl.gov/chart/)



(b) ^{242}Pu



(c) ^{238}U

Figure 2.1: Level schemes of single rotational bands. Spin and parity as indicated, energies in keV. (Band 1 for fitting. From National Nuclear Data Center, URL: http://www.nndc.bnl.gov/chart/)

Table 2.1: Results of NLO fitting for ^{172}Yb, ^{242}Pu and ^{238}U. (Values in keV)

	$a\ (\xi \sim 10)$	$b\ \left(\xi \left(\frac{\xi}{\Omega}\right)^2 \sim 0.001\right)$
^{172}Yb	12.92985	-0.00435
^{242}Pu	7.15542	-0.00175
^{238}U	6.97666	-0.00173

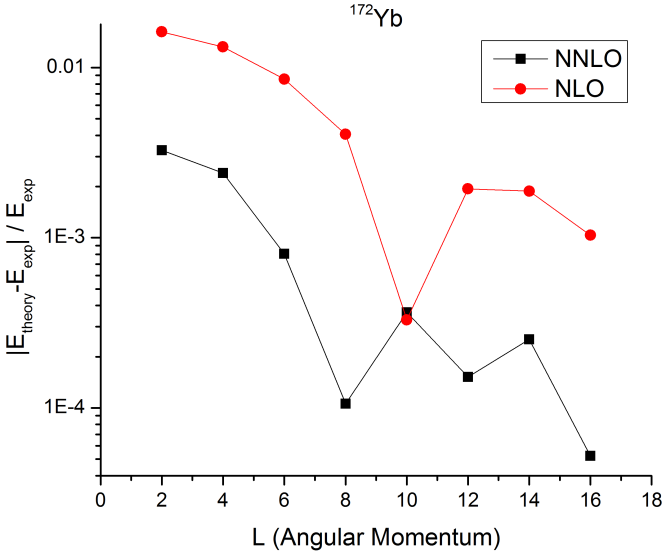
This demonstrates the consistency of our effective theory!

If we insert the obtained a, b and c into E_{NNLO} Eq.2.26, and calculate the energy of each rotational state in Band 1 of ^{172}Yb, ^{242}Pu and ^{238}U, we can compute the relative errors of our effective theory with respect to the experimental values of rotational excitation energy. They are plotted in Figure 2.2 ((a)^{172}Yb, (b)^{242}Pu, (c)^{238}U). Only NLO and NNLO fitting are done for ^{172}Yb and ^{242}Pu rotational bands, whereas ^{238}U band is fitted in all LO, NLO and NNLO. Clearly, NNLO calculation is of the highest accuracy, and then NLO, with LO being the least accurate. Except for one or two singularities, on average the precision is raised by approximately one order of magnitude every time we go one order higher in our effective theory.

Also, it is evident that when the angular momentum is small, the spectra with higher-order corrections are more stably accurate than low-order spectra. When the angular momentum quantum number exceeds a threshold (at around 8 to 12), relative errors of both NLO and NNLO spectra start to slightly fluctuate. This could be due to

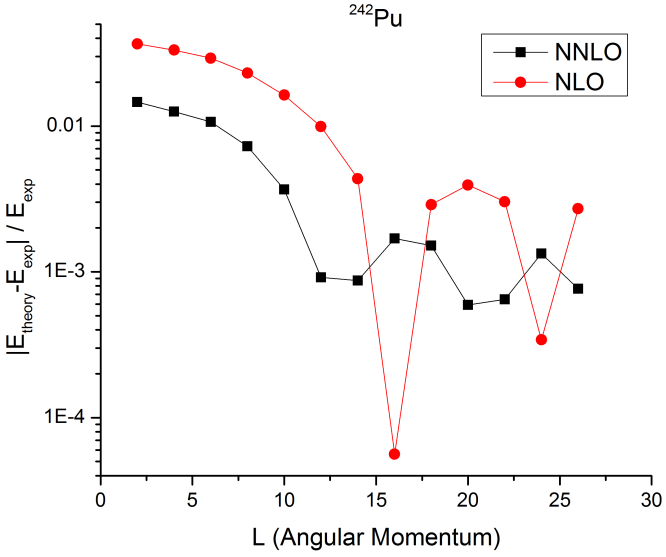
Table 2.2: Results of NNLO fitting for ^{172}Yb, ^{242}Pu and ^{238}U. (Values in keV)

	$a\ (\xi \sim 10)$	$b\ \left(\xi \left(\frac{\xi}{\Omega}\right)^2 \sim 0.001\right)$	$c\ \left(\xi \left(\frac{\xi}{\Omega}\right)^4 \sim 10^{-7}\right)$
^{172}Yb	13.11313	-0.00659	5.93085×10^{-6}
^{242}Pu	7.32346	-0.00256	8.54744×10^{-7}
^{238}U	7.33326	-0.00303	1.04661×10^{-6}



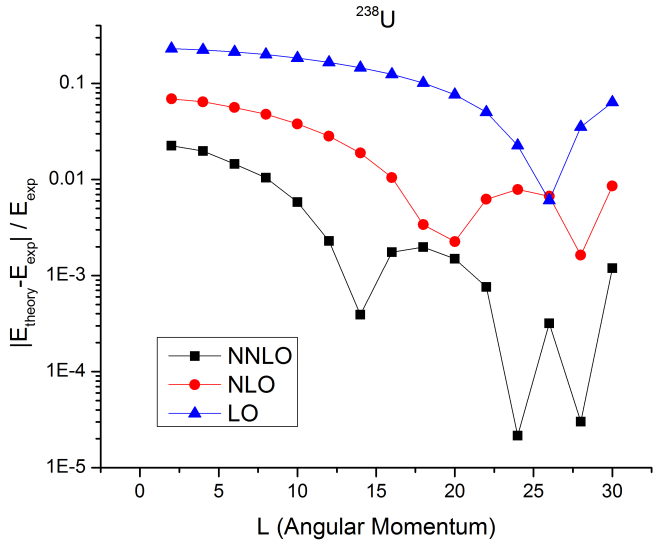
(a) ^{172}Yb

Figure 2.2: Effective theory calculations compared to experiments for single rotational bands



(b) ^{242}Pu

Figure 2.2: Effective theory calculations compared to experiments for single rotational bands



(c) ^{238}U

Figure 2.2: Effective theory calculations compared to experiments for single rotational bands

the omitted physics at the breakdown scale. To be more specific, in the level schemes (Figure 2.1), we see that for each element the excitations of higher-energy degrees of freedom start to come into play at around the energy of 8^+ or 12^+ rotational state. One has to take into consideration the influence of other degrees of freedom (e.g. vibration, pairing etc.), in order to make the higher-order corrections complete and more precise.

If we look at the NNLO spectra of pure Nambu-Goldstone modes Eq.2.20, we could estimate it as

$$E_{\mathrm{NNLO}} \approx \xi + \xi \left(\frac{\xi}{\Omega}\right)^2 + \xi \left(\frac{\xi}{\Omega}\right)^4 \tag{2.28}$$

according to the power counting, which is in essence a Taylor series. We have the even powers of $\frac{\xi}{\Omega}$ as the higher-order corrections, but where are odd powers of it? $\left(\frac{\xi}{\Omega}\right)^1$ is a lower-order correction to ξ and supposed to appear before $\xi\left(\frac{\xi}{\Omega}\right)^2$, similarly for $\left(\frac{\xi}{\Omega}\right)^3$. Strictly speaking, $\xi\left(\frac{\xi}{\Omega}\right)^2$ is not really NLO correction to the collective excitations, but rather just the NLO correction in pure rotational motion. Neither is

$\xi\left(\frac{\xi}{\Omega}\right)^4$ NNLO correction. Once the excitation energy of deformed nuclei reaches the breakdown scale, i.e. where higher-energy degrees of freedom come into effect, one has to consider these degrees of freedom and relabel the orders of all the correction terms and recalculate the energy spectra that include new physics.

The reason that the threshold quantum number where the relative errors begin to fluctuate is around 10 can be accounted for by the follow estimates. The physics overlooked should give rise to at least $\xi\left(\frac{\xi}{\Omega}\right)$ to the LO rotational energy ξ. When the NLO term in Eq.2.26 also reaches $\xi\left(\frac{\xi}{\Omega}\right)$, i.e.

$$\begin{aligned} l(l+1)\xi\left(\frac{\xi}{\Omega}\right)^2 &\sim \xi\left(\frac{\xi}{\Omega}\right) , \\ \Rightarrow \qquad l &\sim \left(\frac{\Omega}{\xi}\right)^{1/2} . \end{aligned} \tag{2.29}$$

it will become comparable to contributions from the omitted physics, thus signifying a breakdown of our effective theory. For deformed nuclei, $\Omega \sim 1$ MeV and $\xi \sim 10$ keV, thus $l \sim 10$.

The issue of $\xi\left(\frac{\xi}{\Omega}\right)$ corrections (namely, the true NLO corrections in the collective excitations) will be addressed after we go into the next chapter, where the quadrupole vibration is introduced as the degrees of freedom at high-energy scale. We will see that $\xi\left(\frac{\xi}{\Omega}\right)$ is, in fact, the order of magnitude of the corrections caused by coupling between vibration and rotation. The corrections will be computed in detail in next chapter.

Chapter 3

Coupling of Quadrupole Vibration to Rotation

In last chapter, we calculate the higher-order corrections in individual rotational bands of axially deformed nuclei, described purely by Nambu-Goldstone modes. The form of the higher-order corrections is in consistency with those derived by other phenomenological models [24] . For highly excited rotational states, the angular momentum becomes sufficiently large that $l \sim \mathcal{O}\big((\frac{\Omega}{\xi})^{1/2}\big)$, so that the rotational states starts to mix with excitations of higher-energy degrees of freedom, such as vibration, pairing, nucleonic excitations and so on. This is clearly visible in the level schemes (Figure 2.1).

Under such circumstances, one can no longer describe all physics solely in terms of Nambu-Goldstone modes. New degrees of freedom must be included. As most commonly occurring type of vibration is quadrupole vibration, quadrupole phonons have been introduced to describe the vibrational degrees of freedom [13]. Taking into account the fact that the Nambu-Goldstone bosons essentially result from the spontaneous symmetry breaking of the entire quadrupole field ψ, namely, they are part of the degrees of freedom that are already included by ψ, one can parameterize the five components of ψ by imposing an general rotation operation $g \in \mathcal{G}/\mathcal{H}$ (coset

$SO(3)/SO(2)$), leaving it only contain the non-Nambu-Goldstone modes. This leads to a transformed quadrupole field ϕ which is totally independent of Nambu-Goldstone modes. The quadrupole field after parameterization is [13]

$$\phi = \begin{pmatrix} \phi_2 \\ 0 \\ \phi_0 \\ 0 \\ \phi_{-2} \end{pmatrix} , \tag{3.1}$$

where ϕ_0 is real, ϕ_2 and ϕ_{-2} are complex and conjugate to each other, i.e. $\phi_{-2} = \phi_2^*$. The field has been parameterized so that $\phi_1 = \phi_{-1} = 0$. These two degrees of freedom have been described by the Nambu-Goldstone fields E_+ and E_-, whose angular momenta along the axial symmetry axis are also $J_z = \pm 1$.

Note that we still have five degrees of freedom in total describing the collective motion as the Bohr-Mottelson model, but we are using different variables. Bohr-Mottelson model uses two deformation parameters and three Euler angles. We have only two Euler angles describing the orientation of the deformed nuclei. This is indeed appropriate for an axially deformed rigid body. Vibrations are described by three degrees of freedom, included in the quadrupole field ϕ. Later we will see in the vibrational spectrum that for quadrupole vibration, one quantum number represents the excitations of β-vibration, and the other two corresponds to γ-vibration. This way of parameterization of quadrupole field and choice of variables is intimately related to the nonlinear realization of the rotational symmetry.

The building blocks for rotationally invariant Lagrangian now become ϕ_0, ϕ_2, ϕ_{-2}, $D_t\phi_0$, $D_t\phi_2$, $D_t\phi_{-2}$, E_+ and E_-. One must also realize that there is a non-zero vacuum expectation value of the field ϕ at the ground state, let it be v, which is not an observable. The zero component of the quadrupole field ϕ_0 is in fact oscillating around v. Therefore, the form of relevant rotationally invariant potential at the leading order should be that of a harmonic oscillator, if one assumes small motion and minimum potential at equilibrium state. The potential can be written as follows in terms of the

quadrupole field [13]

$$\begin{aligned} V_{\mathrm{LO}}(\phi) &= \frac{\omega_0^2}{2}(\phi_0 - v)^2 + \frac{\omega_2^2}{4}\left|\phi_2\right|^2 \\ &= \frac{\omega_0^2}{2}\varphi_0^2 + \frac{\omega_2^2}{4}\left|\phi_2\right|^2 \end{aligned} \tag{3.2}$$

where we have let $\varphi_0 = \phi_0 - v$. Also note $D_t\phi_0 = \partial_t\phi_0 = \dot{\phi}_0$ and v is constant, so

$$\begin{aligned} D_t\phi_0 &= \dot{\phi}_0 \\ \equiv D_t\varphi_0 &= \dot{\varphi}_0 \ . \end{aligned} \tag{3.3}$$

In another word, what really physically matters is φ_0, which is the deviation of ϕ_0 from the expectation value v. The building blocks of Lagrangian ϕ_0 and $D_t\phi_0$ should be replaced by ϕ_0 and $D_t\varphi_0$, respectively. The power counting and corresponding analysis have been established in Ref. [13]

$$\begin{aligned} v \sim \phi_0 &\sim \xi^{-1/2} \ , \\ \varphi_0 \sim \phi_2 &\sim \Omega^{-1/2} \ , \\ D_t\varphi_0 \sim D_t\phi_2 &\sim \Omega^{1/2} \ , \\ \omega_0 \sim \omega_2 &\sim \Omega \ . \end{aligned} \tag{3.4}$$

These scaling relations enable the expectation value of the LO potential to scale as $\langle V_{\mathrm{LO}}\rangle \sim \Omega$. The LO kinetic terms in quadrupole vibration are $\sim \Omega$, too. Furthermore, NLO and NNLO kinetic terms also inevitably enter the Lagrangian. For the potential, we only consider the LO part V_{LO}. This is because NLO potential V_{NLO} has proved to be arising from anharmonic oscillations [13], which is of little interest in our theory.

At NLO, the Hamiltonian and energy spectrum have be thoroughly calculated [13], which is the well-known rotation-vibration band. In this chapter, we go further to NNLO. The Lagrangian that takes into consideration all of the NNLO correction terms constructed from the "building blocks" has been written out [13]. These terms

are attributed to the coupling between rotation and vibration. It is the principle of effective theory, along with the power counting supported by appropriate physical arguments, that naturally leads us to discover how rotation and vibration should interact with each other.

At first glance, the NNLO correction terms seem simple and exhibit a somewhat symmetrical form. However, once one expands it in terms of Nambu-Goldstone modes α and β, long expressions will appear. It turns out that to implement Legendre transformation by brute force in deriving the Hamiltonian is rather tedious. The primary difficulty is not only because the rotation-vibration coupling terms admix Nambu-Goldstone modes and quadrupole fields together, but also because the admixture prevents the quadratic terms from changing into linear after differentiation. This renders the strict inversion from velocities to momenta difficult. One must resort to some perturbative way to accomplish it.

Fukuda's inversion method is perturbative and is employed here to facilitate the Legendre transformation [41]. We will see that Fukuda's method tremendously simplifies the procedures and makes the calculated high-order Hamiltonian computationally tractable. While the eigenvalues of the rotation-vibration energy have been derived in our effective theory, it is straightforward to calculate the NNLO corrections to the energy spectrum by quantum perturbation theory.

In this chapter, we focus on even-even nuclei, which have zero spin at the ground state. Rotation-vibration coupling corrections to odd-mass and odd-odd nuclei can be tackled in a similar way. Since odd nuclei could have non-zero ground state spin, they exhibit some other interesting features that even-even nuclei don't have. Even the LO energy spectrum of odd nuclei is fundamentally different than that of even-even nuclei. These are the topics of next chapter.

3.1 Higher-order Lagrangian

For even-even nuclei, we have following kinetic terms constructed by the quadrupole field ϕ constituting the Lagrangian [13]

$$\begin{aligned}(D_t\varphi_0)^2 &= \dot{\varphi_0}^2 \,, \\ D_t\phi_2 D_t\phi_{-2} &= (\partial_t\phi_2 - 2i\phi_2 E_z)\,(\partial_t\phi_{-2} + 2i\phi_{-2}E_z) \\ &= |\dot{\phi}_2|^2 - 4E_z\mathrm{Im}(\dot{\phi}_2\phi_2^*) + 4|\phi_2^2|E_z^2 \,, \end{aligned} \tag{3.5}$$

A closer inspection to the scale of each term in Eq.3.5 will result in the power counting

$$\begin{aligned}\dot{\varphi_0}^2 \sim |\dot{\phi}_2|^2 &\sim \Omega \,, \\ E_z\mathrm{Im}(\dot{\phi}_2\phi_2^*) \sim E_z &\sim \xi \,, \\ |\phi_2^2|E_z^2 &\sim \xi\,\frac{\xi}{\Omega} \,. \end{aligned} \tag{3.6}$$

Note that the definition of the order of magnitude is different from in last chapter. Here, the leading order is $\sim \mathcal{O}(\Omega)$, the next-to-leading order is $\sim \mathcal{O}(\xi)$, and the next-to-next-to-leading order is $\sim \xi\left(\frac{\xi}{\Omega}\right)$, due to the introduction of high-energy quadrupole vibrations. The second term in $D_t\phi_2 D_t\phi_{-2}$ is of NLO and the third term is of NNLO. Therefore, the LO Lagrangian that describes the quadrupole vibration is [13]

$$L_{\mathrm{LO}} = \frac{1}{2}\dot{\varphi_0}^2 + |\dot{\phi}_2|^2 - \frac{\omega_0^2}{2}\varphi_0^2 - \frac{\omega_2^2}{4}\,|\phi_2|^2 \,. \tag{3.7}$$

This is simply the Lagrangian that describes the harmonic oscillations of an axially symmetrical rigid body in three dimensions. The constant $\frac{1}{2}$ and 1 before the kinetic terms are arbitrary since one can always include them into the field. They are chosen for the convenience to solve the Schrödinger equation.

At NLO (order $\mathcal{O}(\xi)$), the contributions of quadrupole vibration and Nambu-Goldstone modes exist simultaneously in the Lagrangian. The Lagrangian becomes [13]

$$L_{\rm NLO} = L_{\rm LO} + \frac{C_0}{2}(E_+E_-) - 4E_z{\rm Im}(\dot{\phi}_2\phi_2^*) \ . \tag{3.8}$$

Here ϕ_2 can be written as $\phi_2 = \varphi_2 e^{i\gamma}$ in the polar coordinates, where the amplitude φ_2 and direction γ are both real and independent of each other. The NLO Lagrangian is

$$L_{\rm NLO} = L_{\rm LO} + \frac{C_0}{2}(\dot{\alpha}^2\sin^2\beta + \dot{\beta}^2) + 4\varphi_2^2\dot{\gamma}\dot{\alpha}\cos\beta \ , \tag{3.9}$$

$$L_{\rm LO} = \frac{1}{2}\dot{\varphi_0}^2 + \dot{\varphi_2}^2 + \varphi_2^2\dot{\gamma}^2 - \frac{\omega_0^2}{2}\varphi_0^2 - \frac{\omega_2^2}{4}\varphi_2^2 \ . \tag{3.10}$$

The Hamiltonian and eigenenergies have been derived [13]

$$H_{\rm NLO} = H_{\rm LO} + \frac{1}{2C_0}\left(\frac{1}{\sin^2\beta}(p_\alpha - 2\hat{l}_2\cos\beta)^2 + p_\beta^2\right) \ , \tag{3.11}$$

$$E_{\rm NLO}(n_0,n_2,l_2,l) = \omega_0(n_0+\frac{1}{2}) + \frac{\omega_2}{2}(2n_2+|l_2|+1) + \frac{1}{2C_0}\left(l(l+1) - (2l_2)^2\right) \ , \tag{3.12}$$

where $\hat{l}_2$ is defined as $\hat{l}_2 = -i\partial_\gamma$, which is the operator of the azimuthal component of angular momentum. The corresponding quantum number is denoted by l_2.

Indeed, the first two terms in Eq.3.12 represent the eigenenergy of the quadrupole vibration of the axially deformed nuclei. Both are of order $\mathcal{O}(\Omega)$, and (n_0, n_2, l_2) are integer quantum numbers that determines the vibrational state ($n_0 = 0, 1, 2, \cdots$; $n_2 = 0, 1, 2, \cdots$; $l_2 = 0, \pm1, \pm2, \cdots$). The variation of n_0 represents β-vibration excitations corresponding to β-band, and variations of n_2 and l_2 signals excitations of γ-vibration with respect to γ-band. The third term is the eigenenergy of rotational excitations, which is of order $\mathcal{O}(\xi)$. Above each vibrational state there is a rotational band. The moment of inertia is the same for all vibrational states at this order. However, soon we will this is only true at NLO. In NNLO corrections, the moment

of inertia will be altered through the rotation-vibration coupling. We will use the NLO eigenenergy and Hamiltonian to calculate the NNLO spectrum by perturbation theory.

The NNLO Lagrangian that includes all possible coupling ways between rotation and vibration is [13]

$$L_{\rm NNLO} = L_{\rm NLO} + 4|\phi_2|^2 E_z^2 + \Delta L_{\rm NNLO} \ , \tag{3.13}$$

$$\begin{aligned}\Delta L_{\rm NNLO} = {} & D_0(E_+E_-)\varphi_0^2 + F_0(E_+E_-)\dot{\varphi_0}^2 \\ & + D_1\varphi_0(\phi_2 E_-^2 + \phi_{-2}E_+^2) + F_1\dot{\varphi_0}(E_+^2 D_t\phi_{-2} + E_-^2 D_t\phi_{+2}) \\ & + D_2(E_+E_-)|\phi_2|^2 + F_2(E_+E_-)|D_t\phi_2|^2 \ .\end{aligned} \tag{3.14}$$

Each term in $\Delta L_{\rm NNLO}$ has the order of magnitude $\mathcal{O}(\xi^2/\Omega)$, making the undetermined coefficients scale as

$$\begin{aligned} D_0 \sim D_1 \sim D_2 \sim 1 \ , \\ F_0 \sim F_1 \sim F_2 \sim \Omega^{-2} \ . \end{aligned} \tag{3.15}$$

Again, the correctness of the scaling relations should be validated by fitting to the experimental level schemes. Now we substitute E_+ and E_- in Eq.3.14 by Euler angles α and β, and rewrite the entire NNLO Lagrangian

$$L_{\rm NNLO} = L_{\rm NLO} + 4\varphi_2^2\dot{\alpha}^2\cos^2\beta + \Delta L_{\rm NNLO} \ , \tag{3.16}$$

$$\begin{aligned}\Delta L_{\rm NNLO} = {} & D_0(\dot{\alpha}^2\sin^2\beta + \dot{\beta}^2)\varphi_0^2 + F_0(\dot{\alpha}^2\sin^2\beta + \dot{\beta}^2)\dot{\varphi_0}^2 \\ & + D_1\varphi_0\left(2(\dot{\alpha}^2\sin^2\beta - \dot{\beta}^2)\varphi_2\cos\gamma + 4\dot{\alpha}\dot{\beta}\sin\beta\varphi_2\sin\gamma\right) \\ & + F_1\dot{\varphi_0}\Big(2(\dot{\alpha}^2\sin^2\beta - \dot{\beta}^2)(\dot{\varphi_2}\cos\gamma - \varphi_2\dot{\gamma}\sin\gamma) \\ & \qquad + 4\dot{\alpha}\dot{\beta}\sin\beta(\dot{\varphi_2}\sin\gamma + \varphi_2\dot{\gamma}\cos\gamma)\Big) \\ & + D_2(\dot{\alpha}^2\sin^2\beta + \dot{\beta}^2)\varphi_2^2 + F_2(\dot{\alpha}^2\sin^2\beta + \dot{\beta}^2)(\dot{\varphi_2}^2 + \varphi_2^2\dot{\gamma}^2) \ .\end{aligned} \tag{3.17}$$

It is complicated to perform the Legendre transformation rigorously with $\Delta L_{\rm NNLO}$. In the next section we will apply Fukuda's perturbative inversion method and obtain the Hamiltonian. Fukuda's method essentially adopts the idea of perturbation.

3.2 Fukuda's inversion method

Now we introduce Fukuda's inversion method [41] and see how it works for our example. First, let us see what the conjugate momenta are in NNLO,

$$\begin{aligned}
p_{\varphi_0} &= \dot{\varphi_0} + \frac{\partial \Delta L_{\rm NNLO}}{\partial \dot{\varphi_0}} \,, \\
p_{\varphi_2} &= 2\dot{\varphi_2} + \frac{\partial \Delta L_{\rm NNLO}}{\partial \dot{\varphi_2}} \,, \\
p_{\gamma} &= 2\varphi_2^2\dot{\gamma} + 4\varphi_2^2\dot{\alpha}\cos\beta + \frac{\partial \Delta L_{\rm NNLO}}{\partial \dot{\gamma}} \,, \\
p_{\alpha} &= C_0\dot{\alpha}\sin^2\beta + 4\varphi_2^2\dot{\gamma}\cos\beta + 8\varphi_2^2\dot{\alpha}\cos^2\beta + \frac{\partial \Delta L_{\rm NNLO}}{\partial \dot{\alpha}} \,, \\
p_{\beta} &= C_0\dot{\beta} + \frac{\partial \Delta L_{\rm NNLO}}{\partial \dot{\beta}} \,.
\end{aligned} \tag{3.18}$$

To compute the Hamiltonian, the heart procedure in Legendre transformation is to conduct the inversion from velocities to conjugate momenta. As is emphasized previously this is clearly not an easy task in our case, because the velocities mix together in $\Delta L_{\rm NNLO}$ and are quadratic in order.

Fukuda *et al.* developed an perturbative approach to deal with such sort of inversion in Legendre transformation, which was initially intended to tackle chiral symmetry breaking problems in Quantum Chromodynamics and Quantum Electrodynamics [42, 43]. It is soon applied to a wide range of topics, such as equilibrium BCS theory of superconductivity [44], density function theory [41], and discontinuous phase transitions [45]. Essentially, Fukuda's inversion method only counts in the LO terms in conjugate momenta, and treats the higher-order terms

perturbatively by organizing the terms with the same order of magnitude together into one equation. For the generalized formalism of Fukuda's inversion method, the reader can refer to Ref. [41]. Here we will illustrate it through directly applying it to the derivation of NLO Hamiltonian first, and the NNLO Hamiltonian follows next.

We start from a power counting on each term in the right-hand side in Eq.3.18. The LO terms scale as

$$\begin{aligned} p_{\varphi_0} \sim \dot{\varphi}_0 &\sim \Omega^{1/2} , \\ p_{\varphi_2} \sim \dot{\varphi}_2 &\sim \Omega^{1/2} , \\ p_\gamma \sim \varphi_2^2 \dot{\gamma} &\sim 1 , \\ p_\alpha \sim C_0 \dot{\alpha} \sim 4\varphi_2^2 \dot{\gamma} \cos\beta &\sim 1 , \\ p_\beta \sim C_0 \dot{\beta} &\sim 1 . \end{aligned} \tag{3.19}$$

Physically p_{φ_0}, p_{φ_2} are momenta, and p_γ, p_α, p_β are angular momenta. They have the energy scale $\hbar$ (which is ~ 1 in natural units). Also note γ should change as $e^{itE_{\rm LO}}$ with respect to time and $\gamma \sim 1$, so $\dot{\gamma}$ must scale as

$$\dot{\gamma} \sim E_{\rm LO} \sim \Omega . \tag{3.20}$$

which is consistent with Eq.3.19. The scaling of higher-order terms is

$$\begin{aligned}
\frac{\partial \Delta L_{\mathrm{NNLO}}}{\partial \dot{\varphi}_0} &\sim \Omega^{1/2}\left(\frac{\xi}{\Omega}\right)^2 , \\
\frac{\partial \Delta L_{\mathrm{NNLO}}}{\partial \dot{\varphi}_2} &\sim \Omega^{1/2}\left(\frac{\xi}{\Omega}\right)^2 , \\
4\varphi_2^2\dot{\alpha}\cos\beta &\sim \frac{\xi}{\Omega} , \\
\frac{\partial \Delta L_{\mathrm{NNLO}}}{\partial \dot{\gamma}} &\sim \left(\frac{\xi}{\Omega}\right)^2 , \\
8\varphi_2^2\dot{\alpha}\cos^2\beta \sim \frac{\partial \Delta L_{\mathrm{NNLO}}}{\partial \dot{\alpha}} &\sim \frac{\xi}{\Omega} , \\
\frac{\partial \Delta L_{\mathrm{NNLO}}}{\partial \dot{\beta}} &\sim \frac{\xi}{\Omega} .
\end{aligned} \tag{3.21}$$

One must realize that the terms that have the same order in the Lagrangian no longer necessarily lead to the same order of magnitude in momenta. For instance, in p_γ, $4\varphi_2^2\dot{\alpha}\cos\beta$ is of NLO, while in p_α, $4\varphi_2^2\dot{\gamma}\cos\beta$ is of LO. Both of them result from the derivative of the term $4\varphi_2^2\dot{\gamma}\dot{\alpha}\cos\beta$ in the NLO Lagrangian. This is simply because the derivatives are regarding to velocities of different scales (which are $\dot{\alpha}$ and $\dot{\gamma}$ here).

Therefore, to decompose the velocities into components in different orders of magnitude and label them in a systematical way become necessary, as follows

$$\begin{aligned}
\dot{\varphi}_0 &= \dot{\varphi}_0{}^{(0)} + \dot{\varphi}_0{}^{(1)} + \dot{\varphi}_0{}^{(2)} + \dots , \\
\dot{\varphi}_2 &= \dot{\varphi}_2{}^{(0)} + \dot{\varphi}_2{}^{(1)} + \dot{\varphi}_2{}^{(2)} + \dots , \\
\dot{\gamma} &= \dot{\gamma}^{(0)} + \dot{\gamma}^{(1)} + \dot{\gamma}^{(2)} + \dots , \\
\dot{\alpha} &= \dot{\alpha}^{(0)} + \dot{\alpha}^{(1)} + \dot{\alpha}^{(2)} + \dots , \\
\dot{\beta} &= \dot{\beta}^{(0)} + \dot{\beta}^{(1)} + \dot{\beta}^{(2)} + \dots .
\end{aligned} \tag{3.22}$$

It is assumed that $\dot{x}^{(0)}$ has the same order of magnitude with $\dot{x}$ and is of leading order. Higher-order velocities scale their orders of magnitude with the following recursive

relation

$$\dot{x}^{(i+1)} \sim \dot{x}^{(i)} \frac{\xi}{\Omega} \,. \tag{3.23}$$

Note that we need to truncate each line in Eq.3.22 at a certain order, and not each velocity should be truncated at the same order of magnitude. We take the derivation of NLO Hamiltonian as an example first.

$$H_{\rm NLO} = p_{\varphi_0}\dot{\varphi}_0 + p_{\varphi_2}\dot{\varphi}_2 + p_\gamma\dot{\gamma} + p_\alpha\dot{\alpha} + p_\beta\dot{\beta} - L_{\rm NLO} \,. \tag{3.24}$$

The Hamiltonian is accurate up to $\mathcal{O}(\xi)$. Thus, $\dot{\varphi}_0$, $\dot{\varphi}_2$ and $\dot{\gamma}$ should be truncated after the second term, whereas we only need to keep the first term in $\dot{\alpha}$ and $\dot{\beta}$. The Hamiltonian can be rewritten as

$$\begin{aligned} H_{\rm NLO} = {} & p_{\varphi_0}(\dot{\varphi}_0{}^{(0)} + \dot{\varphi}_0{}^{(1)}) + p_{\varphi_2}(\dot{\varphi}_2{}^{(0)} + \dot{\varphi}_2{}^{(1)}) \\ & + p_\gamma(\dot{\gamma}^{(0)} + \dot{\gamma}^{(1)}) + p_\alpha\dot{\alpha}^{(0)} + p_\beta\dot{\beta}^{(0)} - L_{\rm NLO} \,. \end{aligned} \tag{3.25}$$

Furthermore, we need to perform the inversion from Eq.3.18, i.e. express the velocities in terms of the momenta. According to Fukuda's inversion method, only leading-order terms in the right-hand side of Eq.3.18 should be kept for the inversion,

$$\begin{aligned} p_{\varphi_0} &= \dot{\varphi}_0{}^{(0)} \,, \\ p_{\varphi_2} &= 2\dot{\varphi}_2{}^{(0)} \,, \\ p_\gamma &= 2\varphi_2^2\dot{\gamma}^{(0)} \,, \\ p_\alpha &= C_0\dot{\alpha}^{(0)}\sin^2\beta + 4\varphi_2^2\dot{\gamma}^{(0)}\cos\beta \,, \\ p_\beta &= C_0\dot{\beta}^{(0)} \,. \end{aligned} \tag{3.26}$$

According to the principle of perturbation theory, the higher-order terms should satisfy the sets of equation

$$
\begin{aligned}
0 &= \dot{\varphi_0}^{(1)} \ , \\
0 &= 2\dot{\varphi_2}^{(1)} \ , \\
0 &= 2\varphi_2^2\dot{\gamma}^{(1)} + 4\varphi_2^2\dot{\alpha}^{(0)}\cos\beta \ , \\
0 &= C_0\dot{\alpha}^{(1)}\sin^2\beta + 4\varphi_2^2\dot{\gamma}^{(1)}\cos\beta + 8\varphi_2^2\dot{\alpha}^{(0)}\cos^2\beta + \left(\frac{\partial\Delta L_{\mathrm{NNLO}}}{\partial\dot{\alpha}}\right)^{(0)} \ , \\
0 &= C_0\dot{\beta}^{(1)} + \left(\frac{\partial\Delta L_{\mathrm{NNLO}}}{\partial\dot{\beta}}\right)^{(0)} \ .
\end{aligned}
\tag{3.27}
$$

i.e. they are totally negligible compared to the LO terms, but non-negligible between themselves. According to Eq.3.26 and Eq.3.27, the inversion from velocities to momenta is

$$
\begin{aligned}
\dot{\varphi_0}^{(0)} &= p_{\varphi_0} \ , \\
\dot{\varphi_2}^{(0)} &= \frac{1}{2}p_{\varphi_2} \ , \\
\dot{\gamma}^{(0)} &= \frac{p_\gamma}{2\varphi_2^2} \ , \\
\dot{\alpha}^{(0)} &= \frac{1}{C_0\sin^2\beta}(p_\alpha - 2p_\gamma\cos\beta) \ , \\
\dot{\beta}^{(0)} &= \frac{1}{C_0}p_\beta \ .
\end{aligned}
\tag{3.28}
$$

In addition, it is in fact sufficient to use the first three equations in Eq.3.27, because the last two equations about $\dot{\alpha}^{(1)}$ and $\dot{\beta}^{(1)}$ will lead to higher-order terms than NNLO in the Hamiltonian. Concrete forms of the derivatives of ΔL_{NNLO} turn out actually irrelevant to the final Hamiltonian. The velocities in the Lagrangian Eq.3.9, Eq.3.10

should also be replaced with Eq.3.22,

$$\begin{aligned} L_{\mathrm{NLO}} =& \frac{1}{2}\dot{\varphi_0}^2 + \dot{\varphi_2}^2 + \varphi_2^2\dot{\gamma}^2 - \frac{\omega_0^2}{2}\varphi_0^2 - \frac{\omega_2^2}{4}\varphi_2^2 \\ &+ \frac{C_0}{2}(\dot{\alpha}^2\sin^2\beta + \dot{\beta}^2) + 4\varphi_2^2\dot{\gamma}\dot{\alpha}\cos\beta \\ =& \frac{1}{2}(\dot{\varphi_0}^{(0)})^2 + \dot{\varphi_0}^{(0)}\dot{\varphi_0}^{(1)} + (\dot{\varphi_2}^{(0)})^2 + 2\dot{\varphi_2}^{(0)}\dot{\varphi_2}^{(1)} \\ &+ \varphi_2^2\left((\dot{\gamma}^{(0)})^2 + 2\dot{\gamma}^{(0)}\dot{\gamma}^{(1)}\right) - \frac{\omega_0^2}{2}\varphi_0^2 - \frac{\omega_2^2}{4}\varphi_2^2 \\ &+ \frac{C_0}{2}\left((\dot{\alpha}^{(0)})^2\sin^2\beta + (\dot{\beta}^{(0)})^2\right) + 4\varphi_2^2\dot{\gamma}^{(0)}\dot{\alpha}^{(0)}\cos\beta\,. \end{aligned} \tag{3.29}$$

Note that the quadratic terms in the Lagrangian are also truncated up to NLO whenever necessary. Put Eq.3.29 back into Eq.3.25,

$$\begin{aligned} H_{\mathrm{NLO}} =& p_{\varphi_0}(\dot{\varphi_0}^{(0)} + \dot{\varphi_0}^{(1)}) + p_{\varphi_2}(\dot{\varphi_2}^{(0)} + \dot{\varphi_2}^{(1)}) \\ &+ p_\gamma(\dot{\gamma}^{(0)} + \dot{\gamma}^{(1)}) + p_\alpha\dot{\alpha}^{(0)} + p_\beta\dot{\beta}^{(0)} \\ &- \frac{1}{2}(\dot{\varphi_0}^{(0)})^2 - \dot{\varphi_0}^{(0)}\dot{\varphi_0}^{(1)} - (\dot{\varphi_2}^{(0)})^2 - 2\dot{\varphi_2}^{(0)}\dot{\varphi_2}^{(1)} \\ &- \varphi_2^2\left((\dot{\gamma}^{(0)})^2 + 2\dot{\gamma}^{(0)}\dot{\gamma}^{(1)}\right) + \frac{\omega_0^2}{2}\varphi_0^2 + \frac{\omega_2^2}{4}\varphi_2^2 \\ &- \frac{C_0}{2}\left((\dot{\alpha}^{(0)})^2\sin^2\beta + (\dot{\beta}^{(0)})^2\right) - 4\varphi_2^2\dot{\gamma}^{(0)}\dot{\alpha}^{(0)}\cos\beta\,. \end{aligned} \tag{3.30}$$

One can quickly match terms that cancel each other in Eq.3.30. Along with Eq.3.28, the NLO Hamiltonian in terms of position and momentum is

$$\begin{aligned} H_{\mathrm{NLO}} =& \frac{1}{2}p_{\varphi_0}^2 + \frac{1}{4}p_{\varphi_2}^2 + \frac{p_\gamma^2}{4\varphi_2^2} + \frac{\omega_0^2}{2}\varphi_0^2 + \frac{\omega_2^2}{4}\varphi_2^2 \\ &+ \frac{1}{2C_0\sin^2\beta}(p_\alpha - 2p_\gamma\cos\beta)^2 + \frac{1}{2C_0}p_\beta^2\,. \end{aligned} \tag{3.31}$$

Note that p_γ is exactly the same thing as $\hat{l}_2$ in Eq.3.11, both of which are the operator of azimuthal component of angular momentum and quantized as $p_\gamma = \hat{l}_2 = -i\partial_\gamma$. The result of employing Fukuda's method is the same with Eq.3.11, which was obtained non-perturbatively and only truncated at NLO in the last step. This justifies

our motivation to apply Fukuda's method further to H_{NNLO}. The procedures of performing Legendre transformation and computing the Hamiltonian at NLO has been largely simplified.

This is also true for the Hamiltonian at NNLO. We will again see that concrete form of ΔL_{NNLO} does not need to be cared in Legendre transformation. It is unnecessary to write it out explicitly at this moment or to calculate its derivatives. Instead, let us leave it as a whole in calculating the Hamiltonian. Similar to Eq.3.25, we write out H_{NNLO} in terms of the decomposed velocities Eq.3.22 that include one additional order

$$\begin{aligned} H_{\mathrm{NNLO}} = & p_{\varphi_0}(\dot{\varphi_0}^{(0)} + \dot{\varphi_0}^{(1)} + \dot{\varphi_0}^{(2)}) + p_{\varphi_2}(\dot{\varphi_2}^{(0)} + \dot{\varphi_2}^{(1)} + \dot{\varphi_2}^{(2)}) \\ & + p_\gamma(\dot{\gamma}^{(0)} + \dot{\gamma}^{(1)} + \dot{\gamma}^{(2)}) + p_\alpha(\dot{\alpha}^{(0)} + \dot{\alpha}^{(1)}) + p_\beta(\dot{\beta}^{(0)} + \dot{\beta}^{(1)}) - L_{\mathrm{NNLO}} \,. \end{aligned} \tag{3.32}$$

The Lagrangian L_{NNLO} is

$$\begin{aligned} L_{\mathrm{NNLO}} = & \frac{1}{2}(\dot{\varphi_0}^{(0)})^2 + \dot{\varphi_0}^{(0)}\dot{\varphi_0}^{(1)} + \frac{1}{2}(\dot{\varphi_0}^{(1)})^2 + \dot{\varphi_0}^{(0)}\dot{\varphi_0}^{(2)} \\ & + (\dot{\varphi_2}^{(0)})^2 + 2\dot{\varphi_2}^{(0)}\dot{\varphi_2}^{(1)} + (\dot{\varphi_2}^{(1)})^2 + 2\dot{\varphi_2}^{(0)}\dot{\varphi_2}^{(2)} \\ & + \varphi_2^2\left((\dot{\gamma}^{(0)})^2 + 2\dot{\gamma}^{(0)}\dot{\gamma}^{(1)} + (\dot{\gamma}^{(1)})^2 + 2\dot{\gamma}^{(0)}\dot{\gamma}^{(2)}\right) \\ & + \frac{C_0}{2}\left(\left((\dot{\alpha}^{(0)})^2 + 2\dot{\alpha}^{(0)}\dot{\alpha}^{(1)}\right)\sin^2\beta + \left((\dot{\beta}^{(0)})^2 + 2\dot{\beta}^{(0)}\dot{\beta}^{(1)}\right)\right) \\ & + 4\varphi_2^2\dot{\gamma}^{(0)}\dot{\alpha}^{(0)}\cos\beta + 4\varphi_2^2\dot{\gamma}^{(0)}\dot{\alpha}^{(1)}\cos\beta + 4\varphi_2^2\dot{\gamma}^{(1)}\dot{\alpha}^{(0)}\cos\beta \\ & - \frac{\omega_0^2}{2}\varphi_0^2 - \frac{\omega_2^2}{4}\varphi_2^2 + 4\varphi_2^2(\dot{\alpha}^{(0)})^2\cos^2\beta + \Delta L^{(0)}_{\mathrm{NNLO}} \,. \end{aligned} \tag{3.33}$$

Still, only terms up to $\mathcal{O}(\xi^2/\Omega)$ are kept and all the higher-order terms are truncated. A large part of these terms have already been calculated in the NLO case Eq.3.30, thus we simply write them together as H_{NLO} in the final Hamiltonian. What we only need to calculate are those new terms of NNLO which do not appear in H_{NLO}. The

NNLO Hamiltonian becomes

$$\begin{aligned} H_{\mathrm{NNLO}} =& H_{\mathrm{NLO}} + p_{\varphi_0}\dot{\varphi_0}^{(2)} + p_{\varphi_2}\dot{\varphi_2}^{(2)} + p_\gamma\dot{\gamma}^{(2)} + p_\alpha\dot{\alpha}^{(1)} + p_\beta\dot{\beta}^{(1)} \\ & - \frac{1}{2}(\dot{\varphi_0}^{(1)})^2 - \dot{\varphi_0}^{(0)}\dot{\varphi_0}^{(2)} - (\dot{\varphi_2}^{(1)})^2 - 2\dot{\varphi_2}^{(0)}\dot{\varphi_2}^{(2)} \\ & - \varphi_2^2\left((\dot{\gamma}^{(1)})^2 + 2\dot{\gamma}^{(0)}\dot{\gamma}^{(2)}\right) - C_0\dot{\alpha}^{(0)}\dot{\alpha}^{(1)}\sin^2\beta - C_0\dot{\beta}^{(0)}\dot{\beta}^{(1)} \\ & - 4\varphi_2^2\dot{\gamma}^{(0)}\dot{\alpha}^{(1)}\cos\beta - 4\varphi_2^2\dot{\gamma}^{(1)}\dot{\alpha}^{(0)}\cos\beta \\ & - 4\varphi_2^2(\dot{\alpha}^{(0)})^2\cos^2\beta - \Delta L^{(0)}_{\mathrm{NNLO}} \,. \end{aligned} \tag{3.34}$$

Using the equations Eq.3.27 and Eq.3.28, we obtain

$$\begin{aligned} H_{\mathrm{NNLO}} =& H_{\mathrm{NLO}} + p_\alpha\dot{\alpha}^{(1)} - \varphi_2^2(\dot{\gamma}^{(1)})^2 - C_0\dot{\alpha}^{(0)}\dot{\alpha}^{(1)}\sin^2\beta \\ & - 4\varphi_2^2\dot{\gamma}^{(0)}\dot{\alpha}^{(1)}\cos\beta - 4\varphi_2^2\dot{\gamma}^{(1)}\dot{\alpha}^{(0)}\cos\beta \\ & - 4\varphi_2^2(\dot{\alpha}^{(0)})^2\cos^2\beta - \Delta L^{(0)}_{\mathrm{NNLO}} \,. \end{aligned} \tag{3.35}$$

After a few manipulation one finally gets

$$H_{\mathrm{NNLO}} = H_{\mathrm{NLO}} - \Delta L^{(0)}_{\mathrm{NNLO}} \,. \tag{3.36}$$

i.e., except for $\Delta L^{(0)}_{\mathrm{NNLO}}$, all other terms finally cancel each other. We see that the form of $\Delta L^{(0)}_{\mathrm{NNLO}}$ does not really enter our calculation, because the derivatives of it in the conjugate momenta are of higher order than NNLO and can thus be neglected.

Moreover, the resulting NNLO Hamiltonian implies a fact that even if we do not consider any coupling between vibration and rotation, but simply take into account all kinetic terms up to NNLO, we are still able to get the same rotational-vibrational spectrum as obtained by the NLO Hamiltonian. The presence of the kinetic term $4|\phi_2|^2E_z^2$ in the NNLO Lagrangian does not affect the spectrum at all. For completeness we should indeed include it in our Lagrangian, so that each kinetic term constructed from the "building blocks" of the rotationally invariant Lagrangian is taken into consideration.

$\Delta L^{(0)}_{\rm NNLO}$ simply means that each velocity in Eq.3.17 is replaced by the LO inversion using Eq.3.26. In the next section, we will see it is actually even more straightforward not to replace the velocities with momenta when calculating the expectation values with perturbation theory, because terms in $\Delta L^{(0)}_{\rm NNLO}$ either have certain conserved quantities that can be denoted by quantum numbers, or become zero in the first-order perturbation.

3.3 Rotational-vibrational spectrum in next-to-next-to-leading order

We have the NNLO Hamiltonian Eq.3.36, with $\Delta L^{(0)}_{\rm NNLO}$ being

$$\begin{aligned}
\Delta L^{(0)}_{\rm NNLO} &= D_0 \left((\dot\alpha^{(0)})^2 \sin^2\beta + (\dot\beta^{(0)})^2\right) \varphi_0^2 \\
&+ F_0 \left((\dot\alpha^{(0)})^2 \sin^2\beta + (\dot\beta^{(0)})^2\right) (\dot\varphi_0{}^{(0)})^2 \\
&+ D_1\varphi_0 \Big(2\big((\dot\alpha^{(0)})^2 \sin^2\beta - (\dot\beta^{(0)})^2\big)\varphi_2\cos\gamma + 4\dot\alpha^{(0)}\dot\beta^{(0)}\sin\beta\varphi_2\sin\gamma\Big) \\
&+ F_1\dot\varphi_0{}^{(0)}\Big(2\big((\dot\alpha^{(0)})^2 \sin^2\beta - (\dot\beta^{(0)})^2\big)(\dot\varphi_2{}^{(0)}\cos\gamma - \varphi_2\dot\gamma^{(0)}\sin\gamma) \\
&\qquad +4\dot\alpha^{(0)}\dot\beta^{(0)}\sin\beta(\dot\varphi_2{}^{(0)}\sin\gamma + \varphi_2\dot\gamma^{(0)}\cos\gamma)\Big) \\
&+ D_2 \left((\dot\alpha^{(0)})^2 \sin^2\beta + (\dot\beta^{(0)})^2\right) \varphi_2^2 \\
&+ F_2 \left((\dot\alpha^{(0)})^2 \sin^2\beta + (\dot\beta^{(0)})^2\right) \left((\dot\varphi_2{}^{(0)})^2 + \varphi_2^2(\dot\gamma^{(0)})^2\right) \ . \qquad (3.37)
\end{aligned}$$

We need to solve the eigenenergy of the NNLO Hamiltonian, and see what the rotational-vibrational energy spectrum at NNLO is. The basic idea is still perturbation theory.

Let us first look at what the expression $\left((\dot\alpha^{(0)})^2 \sin^2\beta + (\dot\beta^{(0)})^2\right)$ is by substituting $\dot\alpha^{(0)}$ and $\dot\beta^{(0)}$ with Eq.3.26

$$(\dot\alpha^{(0)})^2 \sin^2\beta + (\dot\beta^{(0)})^2 = \frac{1}{C_0^2}\left(\frac{1}{\sin^2\beta}(p_\alpha - 2p_\gamma\cos\beta)^2 + p_\beta^2\right) \ . \qquad (3.38)$$

The total angular momentum Q has been proved to be a conserved quantity for the axially deformed nuclei [13]

$$Q^2 = \frac{1}{\sin^2\beta}(p_\alpha - 2p_\gamma \cos\beta)^2 + p_\beta^2 + (2l_2)^2 \ . \tag{3.39}$$

Therefore Eq.3.38 equals

$$(\dot{\alpha}^{(0)})^2 \sin^2\beta + (\dot{\beta}^{(0)})^2 = \frac{1}{C_0^2}\left(Q^2 - (2l_2)^2\right) \ , \tag{3.40}$$

i.e. no matter how the quantum number l_2 changes, $\left((\dot{\alpha}^{(0)})^2 \sin^2\beta + (\dot{\beta}^{(0)})^2\right)$ will always have the expectation value

$$\begin{aligned} \langle (\dot{\alpha}^{(0)})^2 \sin^2\beta + (\dot{\beta}^{(0)})^2 \rangle &= \frac{1}{C_0^2}\left(Q^2 - (2l_2)^2\right) \\ &= \frac{1}{C_0^2}\left(l(l+1) - (2l_2)^2\right) \ , \end{aligned} \tag{3.41}$$

where l is the quantum number of total angular momentum. Furthermore, as the Nambu-Goldstone modes and quadrupole fields have been parameterized to be independent with each other, we can calculate their correspondent expectation values separately as long as they appear in multiplication. The solutions to φ_0 and φ_2, γ are nothing but the solutions to 1-dimensional and 2-dimensional harmonic oscillators. The eigenenergies of the vibrations in NLO have been given in Eq.3.12. According to quantum harmonic oscillator theory, we have the following expectation values for the velocity and position and their quadratics

$$\langle \varphi_0 \rangle = \langle \dot{\varphi_0}^{(0)} \rangle = 0 \ , \tag{3.42}$$

and

$$\begin{aligned}
\langle\frac{1}{2}\omega_0^2\varphi_0^2\rangle &= \frac{1}{2}\omega_0(n_0+\frac{1}{2}) \,, \\
\langle\frac{1}{2}p_{\varphi_0}^2\rangle &= \frac{1}{2}\omega_0(n_0+\frac{1}{2}) \,, \\
\langle\frac{1}{4}\omega_2^2\varphi_2^2\rangle &= \frac{1}{4}\omega_2(2n_2+|l_2|+1) \,, \\
\langle\frac{1}{4}p_{\varphi_2}^2+\frac{p_\gamma^2}{4\varphi_2^2}\rangle &= \frac{1}{4}\omega_2(2n_2+|l_2|+1) \,.
\end{aligned} \tag{3.43}$$

Thus, the expectation values of the relative terms in the Hamiltonian are

$$\begin{aligned}
\langle\varphi_0^2\rangle &= \frac{1}{\omega_0}(n_0+\frac{1}{2}) \,, \\
\langle(\dot{\varphi_0}^{(0)})^2\rangle &= \omega_0(n_0+\frac{1}{2}) \,, \\
\langle\varphi_2^2\rangle &= \frac{1}{\omega_2}(2n_2+|l_2|+1) \,, \\
\langle(\dot{\varphi_2}^{(0)})^2+\varphi_2^2(\dot{\gamma}^{(0)})^2\rangle &= \frac{1}{4}\omega_2(2n_2+|l_2|+1) \,.
\end{aligned} \tag{3.44}$$

Use Eq.3.44 in Eq.3.37, we immediately obtain the expectation value of $L^{(0)}_{\rm NNLO}$

$$\begin{aligned}
\langle\Delta L^{(0)}_{\rm NNLO}\rangle = \frac{l(l+1)-(2l_2)^2}{C_0^2}\Big(&\frac{D_0}{\omega_0}(n_0+\frac{1}{2})+\frac{D_2}{\omega_2}(2n_2+|l_2|+1) \\
&+F_0\omega_0(n_0+\frac{1}{2})+\frac{1}{4}F_2\omega_2(2n_2+|l_2|+1)\Big) \,.
\end{aligned} \tag{3.45}$$

Finally, it is straightforward for us treat $\Delta L^{(0)}_{\rm NNLO}$ as a small perturbation to $H_{\rm NLO}$, and the first-order perturbation gives us the following NNLO energy spectrum

$$E_{\rm NNLO} = E_{\rm NLO} - \langle\Delta L^{(0)}_{\rm NNLO}\rangle \,. \tag{3.46}$$

Here, $E_{\rm NLO}$ has been given by Eq.3.12. It is trivial to rewrite $-\langle\Delta L^{(0)}_{\rm NNLO}\rangle$ as $\Delta E_{\rm NNLO}$, so that

$$E_{\rm NNLO} = E_{\rm NLO} + \Delta E_{\rm NNLO} \,. \tag{3.47}$$

The NNLO correction to the rotational-vibrational spectrum of deformed nuclei, expressed by Eq.3.45, is the main result of this chapter. In the first-order perturbation, the effect of NNLO correction to the NLO spectrum is just adjusting the moment of inertia by a small amount. While the reciprocal of the original moment of inertia is $\frac{1}{C_0}$, it becomes $\frac{1}{C_0}(1-\epsilon)$ after we consider the rotation-vibration coupling as higher-order corrections for deformed nuclei, where ϵ ($\sim \mathcal{O}(\xi/\Omega)$) equals

$$\begin{aligned}\epsilon &= \frac{2}{C_0}\Big(\frac{D_0}{\omega_0}(n_0+\frac{1}{2})+\frac{D_2}{\omega_2}(2n_2+|l_2|+1) \\ &\qquad +F_0\omega_0(n_0+\frac{1}{2})+\frac{1}{4}F_2\omega_2(2n_2+|l_2|+1)\Big) \\ &= R(n_0+\frac{1}{2})+S(2n_2+|l_2|+1)\ , \end{aligned} \tag{3.48}$$

where R and S are defined as

$$\begin{aligned} R &= \frac{2}{C_0}\left(\frac{D_0}{\omega_0}+F_0\omega_0\right)\ , \\ S &= \frac{2}{C_0}\left(\frac{D_2}{\omega_2}+\frac{1}{4}F_2\omega_2\right)\ . \end{aligned} \tag{3.49}$$

As we see the modification depends on the vibrational band head of the deformed nucleus. In other words, while the moment of inertia of rotation is the same for a particular vibrational state, it is slightly different for different vibrational states. This agrees with a simple view that a deformed nucleus at higher angular velocity should exhibit a centrifugal stretching and thus a larger moment of inertia. There is no reason for nuclei with different deformation to share the same moment of inertia.

Most importantly, most geometric models till now predict a reduced moment of inertia in higher-order corrections to collective excitations [24], whereas our effective theory is more flexible and allows the moment of inertia to either increase or decrease. In fact, it will be shown in the next section that in realistic situations, deformed nuclei at higher vibrational excitation states do have a larger moment of inertia, rather than a reduced one as predicted by the general collective models.

The amount of modification is decided by the vibrational quantum numbers (n_0, n_2, l_2). It is essentially a linear combination of the six constants ω_0, ω_2, D_0, F_0, D_2 and F_2. As in Eq.3.49, they can be redefined into new constants R and S, which can be determined straightforwardly by fitting to data. In addition, though in the first-order perturbation approximation the other two constants D_1 and F_1 do not influence the spectrum, it is no longer the case in the second-order perturbation. This is because φ_0 and $\dot{\varphi}_0$ will both appear as quadratics in the second-order perturbation calculation, which no longer lead to zero expectation values.

3.4 Comparison with level schemes

In this section, we pick several typical level schemes of even-even deformed nuclei to illustrate the effect of NNLO correction to the rotational-vibrational spectrum of form Eq.3.48 derived by our effective theory. We are not going to determine the value of the coefficients by fitting to data, but rather just present a qualitative explanation.

First, let us look at the level scheme of ^{168}Er shown in Figure 3.1. The first three bands are all related to the collective excitations of ^{168}Er. Band 2 is the ground-state band representing the vibrational state $(n_0, n_2, l_2) = (0, 0, 0)$. Band 1 is the β-band at the vibrational excitation state $(1, 0, 0)$, and Band 3 is the γ-band at $(0, 0, 1)$. The Non-band levels are irrelevant for our discussion because they are not collective in nature.

Upon each vibrational state, there is a rotational band. We calculate $(E_{4^+} - E_{2^+})$ for each rotational band and show the results in Table 3.1. Theoretically according to the NLO spectrum in our effective theory $(E_{4^+} - E_{2^+})$ is

$$E_{4^+} - E_{2^+} = \frac{20 - 6}{2C_0} , \tag{3.50}$$

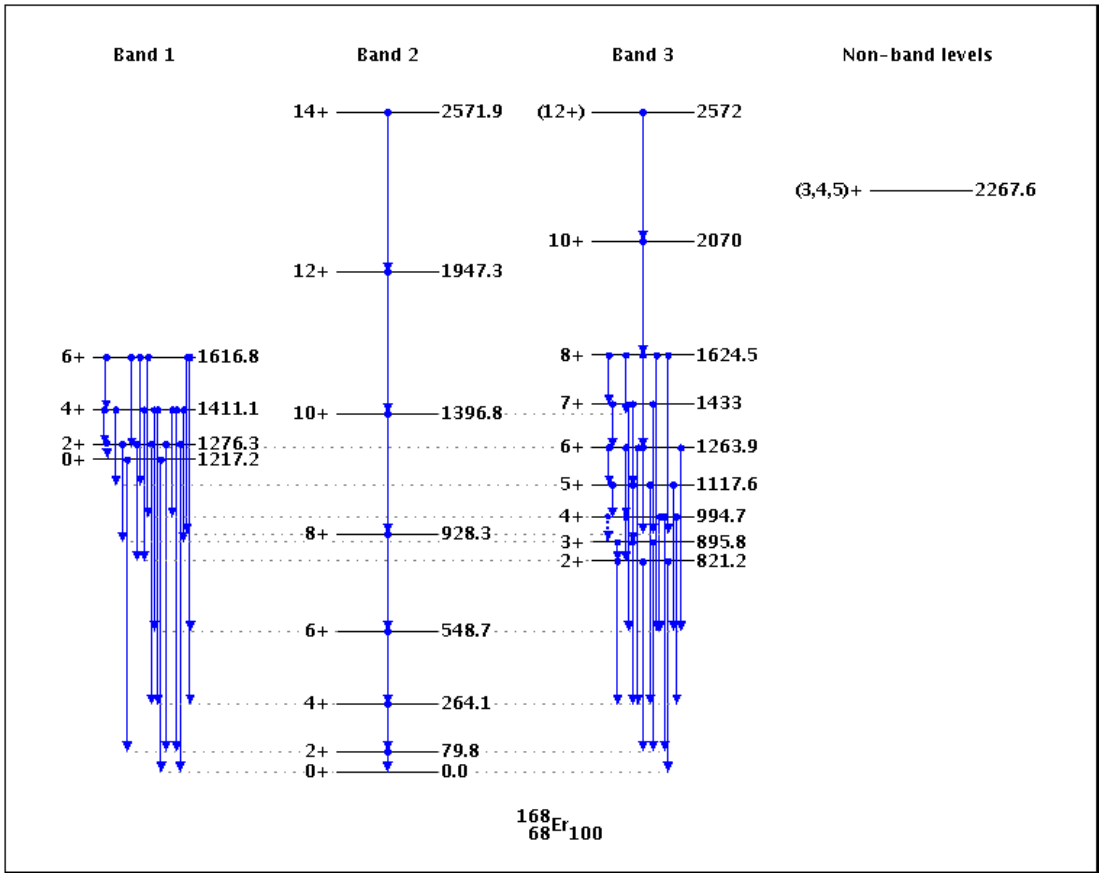


Figure 3.1: Level scheme of ^{168}Er. Spin and parity as indicated, energies in keV. (First three bands. From National Nuclear Data Center, URL: http://www.nndc.bnl.gov/chart/)

Table 3.1: $(E_{4^+} - E_{2^+})$ for first 3 vibrational states of ^{168}Er (in keV)

vibrational state	Band 2 $(0,0,0)$	Band 3 $(0,0,1)$	Band 1 $(1,0,0)$
$E_{4^+} - E_{2^+}$	184.3	173.5	134.8

which is equivalent for each vibrational state.

However, Table 3.1 reveals that different vibrational states do not have the same energy gap between the 4^+ and 2^+ states. This indicates that the C_0 in Eq.3.50, which is proportional to the moment of inertia, is changing among each vibrational state. Moreover, the higher the vibrational excitation energy is, the larger C_0 is, hence the larger the moment of inertia of the deformed nucleus is. This can be explained by our effective theory. Remember C_0 has been shown to be modified approximately as $C_0(1+\epsilon)$, where ϵ is given by Eq.3.48. The moment of inertia predicted by our effective theory indeed has a growing tendency for higher vibrational excitation. The undetermined parameters in ϵ must be positive when taken together as ϵ. If we compare the relative deviation with the power counting estimates (shown in Table 3.2), they are also found to be of the same order of magnitude.

Likewise, a careful examination of ^{162}Dy also uncovers this feature, whose level scheme is shown in Figure 3.2. Band 1 is the ground-state band $(0,0,0)$. Band 2 is the γ-band at the vibrational excitation state (0, 0, 1), and Band 3 is the β-band (1, 0, 0). The values of $(E_{4^+} - E_{2^+})$ are shown in Table 3.3. The comparison of the experimental observation of relative deviation of the moment of inertia at excitation states with

Table 3.2: Relative deviation of moment of inertia at vibrational excitation states from the ground-state value for ^{168}Er

vibrational state	Band 2 $(0,0,0)$	Band 3 $(0,0,1)$	Band 1 $(1,0,0)$
relative deviation	0	6%	27%
$\frac{\xi}{\Omega}$	–	10%	7%

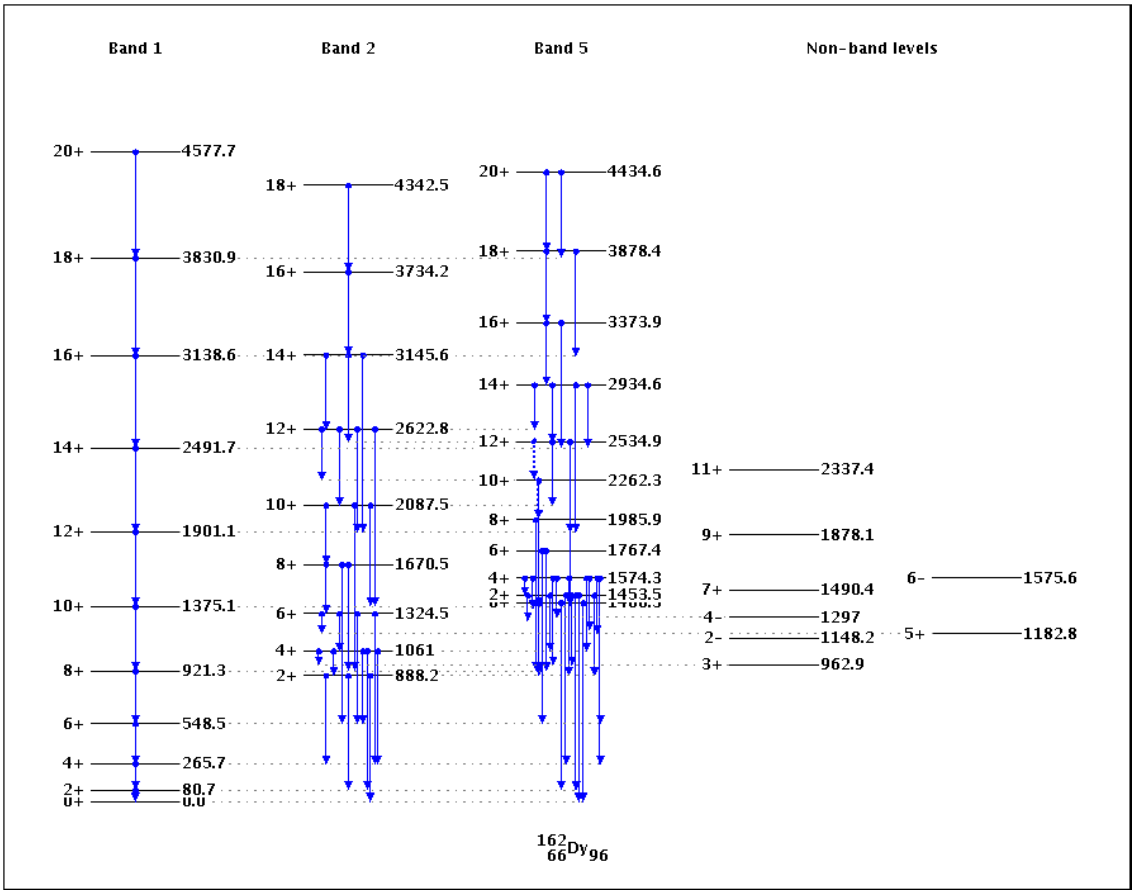


Figure 3.2: Level scheme of ^{162}Dy. Spin and parity as indicated, energies in keV. (First three bands. From National Nuclear Data Center, URL: http://www.nndc.bnl.gov/chart/)

Table 3.3: $(E_{4^+} - E_{2^+})$ for first 3 vibrational states of ^{162}Dy (in keV)

vibrational state	Band 1 $(0,0,0)$	Band 2 $(0,0,1)$	Band 3 $(1,0,0)$
$E_{4^+} - E_{2^+}$	185	172.8	120.8

the power counting estimates is given in Table 3.4. They again demonstrate good agreement.

To pursue a further verification of the power counting Eq.3.15, Eq.3.4 and a higher accuracy, one can numerically fit Eq.3.45 to the data as what has been done in the last chapter.

Till now, the effective theory at NNLO has been studied for even-even nuclei, whose ground states are invariant under time reversal. For odd-mass nuclei, new terms which are not invariant under time reversal will enter the Lagrangian. This makes the rotational-vibrational spectrum distinct from even-even nuclei. They will be studied in next chapter.

Table 3.4: Relative deviation of moment of inertia at vibrational excitation states from the ground-state value for ^{162}Dy

vibrational state	Band 1 $(0,0,0)$	Band 2 $(0,0,1)$	Band 3 $(1,0,0)$
relative deviation	0	7%	35%
$\frac{\xi}{\Omega}$	–	9%	6%

Chapter 4

Nuclei with Finite Ground-state Spins

In last two chapters, the Lagrangians of even-even nuclei are invariant under time reversal at all orders, because even-even nuclei have zero ground-state spin. The minimum value of the angular momentum projection quantum number K in Eq.1.1 is zero, so that the intrinsic spin does not have to be explicitly considered. This also brings about simplicity in constructing the Lagrangian in that the total order of time derivative in any rotationally invariant term built can only be even, thus ruling out the odd-order terms.

Odd-mass nuclei, on the other hand, usually have half-integer spin in their ground state. The minimum value of K in Eq.1.1 is usually 1/2, and spin-orbit coupling requires l must also be half-integer. The ground states of odd nuclei are no longer invariant under time reversal. One of the consequences of the breakdown of time-reversal invariance is, in addition to all the terms already included in even-even nuclei Lagrangian, new terms with odd order of time derivative can come in as time-invariance breaking terms. We have already seen that in the Nambu-Goldstone modes of odd nuclei, a Wess-Zumino term that consists of one time derivative appears in the Lagrangian [13]. It contains a quantum number q that characterizes the spin state.

The resulting rotational bands of odd nuclei are exactly of the form Eq.1.1 with the q being K.

In this chapter, we take a closer look at odd nuclei in our effective theory by adding new rotationally invariant terms into the Lagrangian and seeing how they change the rotational-vibrational spectrum. These terms are related to the quadrupole vibrational degrees of freedom and are not invariant under time reversal. The individual Nambu-Goldstone modes that lead to the rotational band of odd nuclei have been studied [13]. Essentially, we are studying odd deformed nuclei at NLO that couples the quadrupole vibrations to Nambu-Goldstone modes. The NNLO corrections resulting from rotation-vibration interaction are concerned here, which can be investigated in a similar way as in even-even nuclei. We will see the collective excitations of odd nuclei even at NLO displays many interesting properties that even-even nuclei do not have.

4.1 Next-to-leading-order Lagrangian

In odd nuclei, terms that contain odd order of time derivative in total can enter the Lagrangian. In terms of the quadrupole fields, the simplest terms one can construct from the "building blocks" of rotationally invariant Lagrangian are $\phi_0 D_t \phi_0$, $\phi_2 D_t \phi_{-2}$ and $\phi_{-2} D_t \phi_2$. We primarily study the effects of these terms in the following, whereas the results of pure Nambu-Goldstone modes for odd nuclei will also be used.

In last chapter, ϕ_2 is decomposed into a radial part and an angular part in a polar coordinate system. It turns out easier to decompose it on Cartesian coordinates here (mostly due to the simplicity in gauge transformation which we will see in next section).

$$\begin{aligned} \phi_2 &= x + iy \,, \\ \phi_{-2} &= x - iy. \end{aligned} \tag{4.1}$$

Thus,

$$\begin{aligned}
\phi_2 D_t \phi_{-2} &= x\dot{x} + y\dot{y} - i(x\dot{y} - y\dot{x}) + 2iE_z(x^2 + y^2) \ , \\
\phi_{-2} D_t \phi_2 &= x\dot{x} + y\dot{y} + i(x\dot{y} - y\dot{x}) - 2iE_z(x^2 + y^2) \ , \\
\phi_0 D_t \phi_0 &= \phi_0 \dot{\phi}_0 = \frac{1}{2}\partial_t(\phi_0^2) \ . \qquad (4.2)
\end{aligned}$$

Only the linear combinations of $\phi_2 D_t \phi_{-2}$ and $\phi_{-2} D_t \phi_2$ will yield purely real value. A power counting on them gives

$$\phi_2 D_t \phi_{-2} \sim \phi_{-2} D_t \phi_2 \sim \phi_0 D_t \phi_0 \sim 1 \ . \qquad (4.3)$$

Referring to Eq.2.7, we immediately write out the NLO Lagrangian that includes Nambu-Goldstone modes. For simplicity the trivial potential is ignored temporarily until we obtain the final Hamiltonian.

$$\begin{aligned}
L_{\rm LO}^{(oo)} &= (D_t\phi_2)(D_t\phi_{-2}) + \frac{1}{2}\dot{\phi_0}^2 + \frac{A}{2}\partial_t(\phi_0^2) \\
&\quad + \frac{B}{2}\left(\phi_2 D_t \phi_{-2} + \phi_{-2} D_t \phi_2\right) + \frac{i\widetilde{B}}{2}\left(\phi_2 D_t \phi_{-2} - \phi_{-2} D_t \phi_2\right) \ , \qquad (4.4) \\
L_{\rm NLO}^{(oo)} &= L_{\rm LO}^{(oo)} + \frac{C_0}{2}(\dot{\alpha}^2 \sin^2\beta + \dot{\beta}^2) - q\dot{\alpha}\cos\beta \ . \qquad (4.5)
\end{aligned}$$

The term $\frac{A}{2}\partial_t(\phi_0^2)$ is a total time derivative which can be removed from the Lagrangian. However, we keep it at this moment and will later see that it can also be eliminated by a gauge transformation, without entering the final results. Apparently $\phi_2 D_t \phi_{-2}$ and $\phi_{-2} D_t \phi_2$ are also related to the quadrupole vibrational degrees of freedom. Implicitly they can involve other degrees of freedom at the same energy scale such as spin, pairing etc. (In the final results we will see it's more likely to be connected with spin). B, $\widetilde{B}$ and A must scale as

$$B \sim \widetilde{B} \sim A \sim \Omega \ . \qquad (4.6)$$

Note that although $(D_t\phi_2)(D_t\phi_{-2})$, $\phi_2 D_t\phi_{-2}$ and $\phi_{-2}D_t\phi_2$ are all of LO, after being expanded in either polar coordinates or Cartesian coordinates, all of them will produce NLO and NNLO kinetic terms, as in even-even nuclei previously. The Lagrangian after the expansion is

$$\begin{aligned} L_{\rm LO}^{(oo)} &= (\dot{x}^2+\dot{y}^2)+\frac{1}{2}\dot{\phi_0}^2+\frac{A}{2}\partial_t(\phi_0^2) \\ &\quad +B(x\dot{x}+y\dot{y})+\widetilde{B}(x\dot{y}-y\dot{x})\ , \qquad (4.7) \\ L_{\rm NLO}^{(oo)} &= L_{\rm LO}^{(oo)}-4E_z(x\dot{y}-y\dot{x})-2\widetilde{B}(x^2+y^2)E_z \\ &\quad +\frac{C_0}{2}(\dot{\alpha}^2\sin^2\beta+\dot{\beta}^2)-q\dot{\alpha}\cos\beta+\cancel{4(x^2+y^2)E_z^2}\ . \qquad (4.8) \end{aligned}$$

The last term in Eq.4.8 can be neglected because it is of NNLO. One can identify that $x\dot{y}-y\dot{x}$ is the angular momentum equivalent to $\varphi_2^2\dot{\gamma}$ in the polar coordinates. Therefore, except for the terms that contain B or $\widetilde{B}$, everything is exactly the same with the NLO Lagrangian Eq.3.9 and Eq.3.10 of even-even nuclei. We are mainly interested here in the terms with the constant B or $\widetilde{B}$.

4.2 Rotational-vibrational spectrum in next-to-leading order

To derive the Hamiltonian, we again employ Fukuda's inversion method. We start from the conjugate momenta

$$\begin{aligned} p_0 &= \dot{\phi}_0 + A\phi_0 \,, \\ p_x &= 2\dot{x} + Bx - \widetilde{B}y - 4y\dot{\alpha}\cos\beta \,, \\ p_y &= 2\dot{y} + By + \widetilde{B}x + 4x\dot{\alpha}\cos\beta \,, \\ p_\alpha &= 4\cos\beta(x\dot{y} - y\dot{x}) + 2\widetilde{B}\cos\beta(x^2+y^2) \\ &\quad + C_0\dot{\alpha}\sin^2\beta - q\cos\beta \,, \\ p_\beta &= C_0\dot{\beta} \,. \end{aligned} \tag{4.9}$$

The LO approximations for them are

$$\begin{aligned} p_0 &= \dot{\phi}_0^{(0)} + A\phi_0 \,, \\ p_x &= 2\dot{x}^{(0)} + Bx - \widetilde{B}y \,, \\ p_y &= 2\dot{y}^{(0)} + By + \widetilde{B}x \,, \\ p_\alpha &= 4\cos\beta(x\dot{y}^{(0)} - y\dot{x}^{(0)}) + 2\widetilde{B}\cos\beta(x^2+y^2) \\ &\quad + C_0\dot{\alpha}^{(0)}\sin^2\beta - q\cos\beta \,, \\ p_\beta &= C_0\dot{\beta}^{(0)} \,, \end{aligned} \tag{4.10}$$

with the NLO approximations being

$$\begin{aligned}
0 &= \dot{\phi_0}^{(1)} , \\
0 &= 2\dot{x}^{(1)} - 4y\dot{\alpha}^{(0)} \cos\beta, \\
0 &= 2\dot{y}^{(1)} + 4x\dot{\alpha}^{(1)} \cos\beta, \\
0 &= 4\cos\beta(x\dot{y}^{(1)} - y\dot{x}^{(1)}) + C_0\dot{\alpha}^{(1)} \sin^2\beta, \\
0 &= C_0\dot{\beta}^{(1)} .
\end{aligned} \tag{4.11}$$

From Eq.4.10, Fukuda's inversion yields

$$\begin{aligned}
\dot{\phi_0}^{(0)} &= p_0 - A\phi_0 , \\
\dot{x}^{(0)} &= \frac{1}{2}(p_x - Bx + \widetilde{B}y) , \\
\dot{y}^{(0)} &= \frac{1}{2}(p_y - By - \widetilde{B}x) , \\
\dot{\alpha}^{(0)} &= \frac{1}{C_0 \sin^2\beta}\big(p_\alpha - 4\cos\beta(x\dot{y}^{(0)} - y\dot{x}^{(0)}) \\
&\qquad -2\widetilde{B}\cos\beta(x^2+y^2) + q\cos\beta\big) , \\
\dot{\beta}^{(0)} &= \frac{1}{C_0}p_\beta .
\end{aligned} \tag{4.12}$$

A further substitution in $\dot{\alpha}^{(0)}$ simplifies it into

$$\dot{\alpha}^{(0)} = \frac{1}{C_0 \sin^2\beta}\Big(p_\alpha + \big(q - 2(xp_y - yp_x)\big)\cos\beta\Big) . \tag{4.13}$$

The Lagrangian truncated up to NLO is

$$\begin{aligned} L^{(oo)}_{\rm NLO} &= \frac{1}{2}\big(\dot{\phi_0}^{(0)}\big)^2 + A\phi_0\dot{\phi_0}^{(0)} \\ &+ \Big(\big(\dot{x}^{(0)}\big)^2 + \big(\dot{y}^{(0)}\big)^2 + 2\dot{x}^{(0)}\dot{x}^{(1)} + 2\dot{y}^{(0)}\dot{y}^{(1)}\Big) \\ &+ B(x\dot{x}^{(0)} + x\dot{x}^{(1)} + y\dot{y}^{(0)} + y\dot{y}^{(1)}) \\ &+ \widetilde{B}(x\dot{y}^{(0)} + x\dot{y}^{(1)} - y\dot{x}^{(0)} - y\dot{x}^{(1)}) \\ &+ 4\dot{\alpha}^{(0)}\cos\beta(x\dot{y}^{(0)} - y\dot{x}^{(0)}) + 2\widetilde{B}\dot{\alpha}^{(0)}\cos\beta(x^2+y^2) \\ &+ \frac{C_0}{2}\big((\dot{\alpha}^{(0)}\big)^2\sin^2\beta + \big(\dot{\beta}^{(0)})^2\big) - q\dot{\alpha}^{(0)}\cos\beta\,. \end{aligned} \tag{4.14}$$

Then, the NLO Hamiltonian is

$$\begin{aligned} H^{(oo)}_{\rm NLO} &= p_0(\dot{\phi_0}^{(0)} + \dot{\phi_0}^{(1)}) + p_x(\dot{x}^{(0)} + \dot{x}^{(1)}) + p_y(\dot{y}^{(0)} + \dot{y}^{(1)}) \\ &\quad + p_\alpha\dot{\alpha}^{(0)} + p_\beta\dot{\beta}^{(0)} - L^{(oo)}_{\rm NLO} \\ &= \frac{1}{2}(p_0 - A\phi_0)^2 + \frac{1}{4}(p_x - Bx + \widetilde{B}y)^2 + \frac{1}{4}(p_y - By - \widetilde{B}x)^2 \\ &\quad + \frac{1}{2C_0}\left(\frac{1}{\sin^2\beta}\Big(p_a + \big(q - 2(xp_y - yp_x)\big)\cos\beta\Big)^2 + p_\beta^2\right)\,. \end{aligned} \tag{4.15}$$

Note that $xp_y - yp_x$ is the operator of orbital angular momentum in the z-axis, which is just p_γ in Eq.3.31. It can be concluded firstly that the Hamiltonian that describes the rotation is essentially the same with Eq.3.31, except that the quantum number q derived from the Wess-Zumino term $L_{\rm WZ}$ is superimposed onto the azimuthal angular momentum quantum number associated with the quadrupole vibration. As q is the spin of the band head, this effect is simply equivalent to a superposition of the intrinsic angular momentum and orbital angular momentum in the same direction. The introducing of quadrupole degrees of freedom into odd deformed nuclei not only leads to vibrational bands, but also couples the vibration to the intrinsic spin to some extent.

Finally we look at the kinetic terms in the Hamiltonian that describes the

quadrupole vibration, which is

$$H_{\rm VB}^{(oo)} = \frac{1}{2}(p_0 - A\phi_0)^2 + \frac{1}{4}(p_x - Bx + \widetilde{B}y)^2 + \frac{1}{4}(p_y - By - \widetilde{B}x)^2 \ . \tag{4.16}$$

A phase function

$$\lambda(x, y, \phi_0) = \frac{B}{2}(x^2 + y^2) + \frac{A}{2}\phi_0^2 \ , \tag{4.17}$$

can be defined to gauge away unnecessary parameters in the Hamiltonian, so that

$$\vec{\nabla}\lambda = (Bx,\ By,\ A\phi_0) \ . \tag{4.18}$$

Implementing the gauge transformation on Eq.4.16 with Eq.4.18 yields

$$H_{\rm vb}^{(oo)} = \frac{1}{2}p_0^2 + \frac{1}{4}(p_x + \widetilde{B}y)^2 + \frac{1}{4}(p_y - \widetilde{B}x)^2 \ . \tag{4.19}$$

Rewriting $\widetilde{B}$ as ω', expanding the quadratics and putting back the potential Eq.3.2 , we have

$$\begin{aligned} H_{\rm vb}^{(oo)} &= \frac{1}{2}p_0^2 + \frac{1}{4}p_x^2 + \frac{1}{4}p_y^2 \\ &- \frac{1}{2}\omega'(xp_y - yp_x) + \frac{1}{4}\omega'^2(x^2 + y^2) \\ &+ \frac{1}{2}\omega_0^2\varphi_0^2 + \frac{1}{4}\omega_2^2(x^2 + y^2) \ . \end{aligned} \tag{4.20}$$

Recall that $xp_y - yp_x$ is azimuthal angular momentum operator and equal to p_γ or l_2. The quadratic term of ω' alters the potential in the x-y plane, with the angular frequency becoming $(\omega'^2 + w_2^2)^{1/2}$ instead of w_2.

The Hamiltonian Eq.4.20 yields the vibrational eigenenergy

$$E_{\rm vb}^{(oo)} = \omega_0(n_0 + \frac{1}{2}) + \frac{\sqrt{\omega'^2 + \omega_2^2}}{2}(2n_2 + |l_2| + 1) - \frac{1}{2}\omega' l_2 \ . \tag{4.21}$$

If we further incorporate the rotational Hamiltonian with Eq.4.15, we come to the main result of this chapter, namely the energy spectrum

$$\begin{aligned} E_{\rm vb}^{(oo)} &= \omega_0(n_0+\frac{1}{2})+\frac{\sqrt{\omega'^2+\omega_2^2}}{2}(2n_2+|l_2|+1)-\frac{1}{2}\omega' l_2 \\ &\quad +\frac{1}{2C_0}\left(l(l+1)-(q-2l_2)^2\right) \ . \end{aligned} \tag{4.22}$$

This spectrum can be regarded as the rotational-vibrational spectrum of odd nuclei. Compared to the spectrum of even-even nuclei, there are two additional characters in the odd nuclei bands. In the rotational band, the occurrence of finite ground-state spin gives rise to a coupling between the spin and the azimuthal angular momentum associated with the vibration. In the vibrational band, while the angular frequency of the potential is enlarged by a constant, the spacing between adjacent levels is compressed. The degree of compression linearly depends on the azimuthal quantum number l_2.

If we remove the harmonic oscillator potential, the energy spectrum becomes

$$E_{\rm landau}^{(oo)} = \frac{1}{2}\omega'(2n_2+|l_2|-l_2+1)+\frac{1}{2C_0}\left(l(l+1)-(q-2l_2)^2\right) \ . \tag{4.23}$$

The vibrational band is nothing but the well-known Landau levels. The physical meaning of ω' is that it resembles the magnitude of a constant magnetic field acting on the deformed nuclei. This is not surprising because ω' is a constant in front of the one time-derivative term, i.e. velocity, in the Lagrangian. An interaction that depends on velocities is a Lorentz force caused by magnetic fields. The gauge transformation performed in the calculation designates a different choice of the vector potential, which is ensured by the gauge invariance of magnetic field. Hence, ω' represents a measure of the magnetic field.

While there is no external magnetic field exerted on the deformed nucleus, an internal magnetic field can arise from the orbital motion of nucleons around nucleus. For odd nuclei there is always an excess nucleon whose spin or orbital motion is not

offset by others, thus inducing a charged orbital motion of the center of mass of the nucleus.

4.3 Comparison with level schemes

Let us take a look at the experimental data of odd-mass nuclei. Again, only qualitative analysis is given here. The level scheme of ^{241}Pu is shown in Figure 4.1, and of ^{237}U in Figure 4.2.

For ^{241}Pu, the spacings between the band heads of the ground-state band (Band 1) and the lowest two vibrational excitation band (Band 2 and Band 3) are separate by only about 160 keV, which is small in contrast to ~ 1 MeV in even-even nuclei. The implication is the energy gaps between adjacent vibrational states is compressed.

This property is even more clear in the level scheme of ^{237}U, where the spacings between all the adjacent vibrational excitation states in the four bands are around a few tens of keV. The exhibition of high density of states in the vibrational spectrum is indeed a feature of Landau levels.

Furthermore, in the Lagrangian of odd-mass nuclei, the time-odd terms are in the leading order, which means the order of magnitude of their corrections is of the same order with the quadrupole vibration. According to the level schemes of even-even nuclei, the first vibrational excitation energies of ^{240}Pu and ^{236}U are at about 700 keV. In the level schemes of ^{241}Pu and ^{236}U, the lowest-lying vibrational excitation energies range from tens of keV to 200 keV, which differ with their even-even counterparts by hundreds of keV. Indeed, the correction is of the same order of magnitude with the vibrational excitation.

Our effective theory at NLO successfully accounts for some features of odd nuclei. However, there are still a number of properties in the odd nuclei level schemes that are not yet explained by the effective theory. For instance, Band 2 in the level scheme of ^{241}Pu (Figure 4.1) manifests a conspicuous splitting of the energy levels into two branches. This effect, called signature splitting, is a widely observed phenomenon in

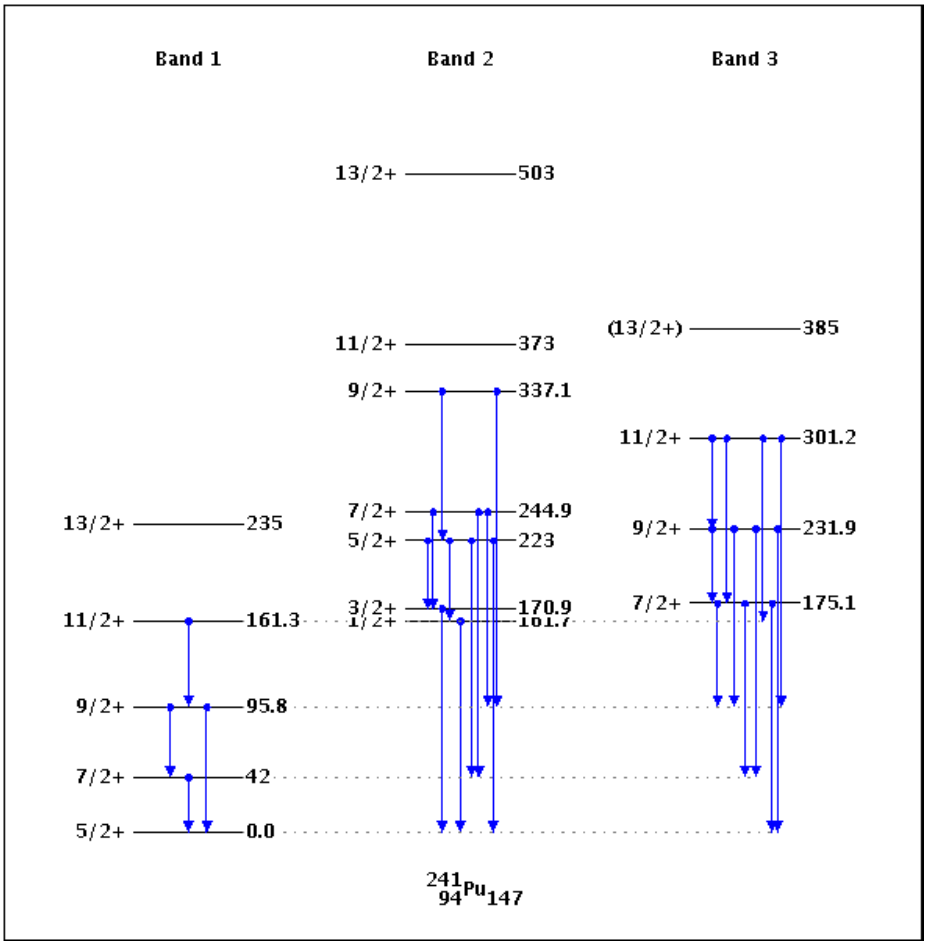


Figure 4.1: Level scheme of ^{241}Pu. Spin and parity as indicated, energies in keV. (From National Nuclear Data Center, URL: www.nndc.bnl.gov/chart/)

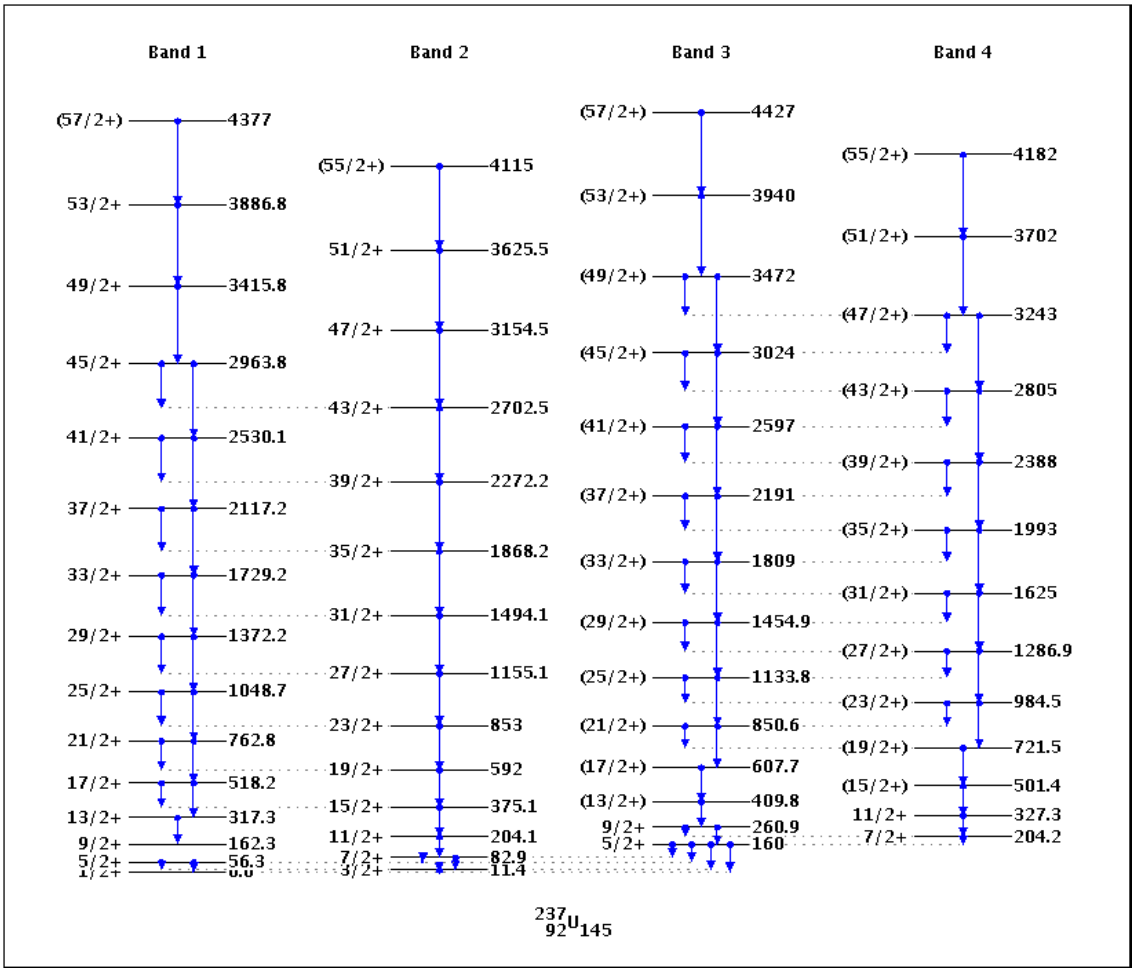


Figure 4.2: Level scheme of ^{237}U. Spin and parity as indicated, energies in keV. (From National Nuclear Data Center, URL: www.nndc.bnl.gov/chart/)

rotational bands of odd-mass deformed nuclei [46, 47], and has been explained by the deformed shell model (Nilsson model) [48].

Signature is a unique quantum number appearing in a deformed intrinsic system [49]. It also has to do with a non-adiabatic correction, known as Coriolis correction, in the rotational bands of odd-mass nuclei. The Coriolis correction is especially significant and strong in band heads with spin $q = 1/2$, and very commonly makes different rotational states mix together. This is still missed in our effective theory.

Chapter 5

Conclusions

To summarize, we have three major results following the high-order calculations in the effective theory for deformed nuclei.

First, the NNLO corrections in Nambu-Goldstone modes is three powers of the LO energy of the rotational state and yields higher accuracy in fitting with experimental level schemes. More importantly, the order of magnitude of all the fitting parameters proves to agree with the power counting estimates of our effective theory, and hence partially validates our effective theory.

Second, the vibrations are coupled to the rotation at NNLO for even-even nuclei, with the interaction between them being fully considered. As the eigenenergy of the NLO effective Hamiltonian reproduces the well-known rotational-vibrational spectrum derived by geometric collective models, the coupling of rotation and vibration at NNLO creates a new effect. The moment of inertia of excited vibrational states is able to deviate from ground-state value not only incrementally, but also decrementally. While the geometric models predict a decreased moment of inertia at higher excitation states, nature often puts it the other way. The amount of deviation depends linearly on the quantum numbers specifying the vibrational band head, with the scaling factors to be determined by fitting to data. The order of magnitude of the deviation in experiments also confirms the estimates of power counting.

Finally, odd-mass nuclei with finite ground-state spin are studied at NLO. Time-odd terms that break the time-reversal symmetry can appear in the Lagrangian at this order. This can be viewed as describing the interaction between the deformed nuclei and a uniform magnetic field. For this reason, the resulting Hamiltonian appears to have gauge fields. After superfluous parameters are gauged away, the time-odd terms ultimately result in a correction term linear with the azimuthal quantum number in the vibrational energy spectrum. The spacing between adjacent vibrational states can be compressed by this correction, hence the spectrum exhibits a high density of states in resemblance to Landau levels. This feature is visible in the experimental level schemes of odd-mass nuclei. The rotational bands are similar to even-even nuclei, except that the non-zero spin is coupled to the azimuthal angular momentum of the quadrupole vibration.

Future interesting problems in the effective theory of deformed nuclei include the study of terms involving second time-derivatives in Nambu-Goldstone modes, the coupling of electromagnetic fields to Nambu-Goldstone modes, the coupling of fermions associated with even higher-energetic degrees of freedom to Nambu-Goldstone modes, and an effective field theory for deformed nuclei. As atomic nuclei cover such rich phenomena and the effective theory provides us with a model-independent description of them, both new challenges and new discoveries await.

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Appendices

Appendix A

Higher-order Time Derivatives

In the previous chapters, all of the Lagrangians contain only the first-order derivatives with respect to time. We know from the classical mechanics that to describe the motion of a classical particle, position and velocity, i.e. time derivatives up to first order, are sufficient. As long as all the positions and velocities are simultaneously specified at a certain instant, the state of the system is considered as completely determined in the sense that the subsequent state can be predicted [50]. However, a thorough examination of all the possible combinations of the "building blocks" of a rotationally invariant Lagrangian in our effective theory inevitably invokes one's curiosity about terms such as $(D_t E_+)(D_t E_-)$, $\phi_2 D_t^2 \phi_{-2}$ and so on. These terms were not explored in previous chapters. They involve the second-order time derivatives of either Nambu-Goldstone modes or quadrupole phonons.

Lagrangians involving second- or higher-order time derivatives have been intensively studied in literatures. Generally, they are present in a wide variety of physical problems as corrections to low-order-derivative theories. In general relativity, quantum corrections that take the form of quadratics in curvature naturally contain higher-order time derivatives of the metric [51, 52]. In the theory of cosmic string, higher-order time derivatives occurs in the correction terms that arise from the effective action of string motion over a curved world sheet and depend on the "rigidity"

of the string [53, 54]. Another well-known example of theories with higher-order time derivative is Dirac's relativistic model of radiating electron [55].

Although high-order derivatives mostly appear as high-order corrections to a low-order derivative theory, the ultimate consequences of the corrections are beyond physicists' original expectation. The high-order time derivative theories are subject to many undesirable features, such as the absence of a low-energy bound, lack of convergence in the limit of an infinitesimal coefficient, etc.. It can be summarized from previous chapters that whenever a first-order time derivative term with a coefficient scaling it is added to the effective Lagrangian as a small correction, the energy will simply be perturbed by a small amount. The perturbation tends to be zero in the limit that the scaling coefficient infinitely approaches zero. However, one can no longer make this assumption and naively treat high-order time derivative terms added to the Lagrangian in this way. The intuition that a high-order derivative term with a small coefficient in front of it acts only as a perturbative correction to original solution is incorrect, however small the coefficient is [56]. This unwelcome property is because the new degrees of freedom introduced by high-order derivatives are unconstrained, which is missing from a low-order derivative theory. A systematic method called "perturbative constraints" has been developed to resolve this issue suffered by high-order derivative theories [57]. It has been successfully applied to a number of cases (Dirac's electron, general relativity, cosmic string etc.) to avoid "runaway" solutions, and render these high-order derivative theories self-consistent [56].

In addition to "perturbative constraints", there is another class of high-order derivative theories – nonlocal theories – that does not suffer the abovementioned problems. Nonlocality usually occurs in low-energy effective theories, where the effective action naturally contains higher-order derivatives resulting from time-retarded interactions. It is recognized to be a fundamental property that all string field theories possess. Ref. [58] has a very nice investigation and discussion about nonlocality in string theory. It is also realized that a truncated series expansion (with higher derivatives) of a nonlocal theory, with perturbative constraints explicitly

imposed, ends up sharing the same solutions with the original full nonlocal theory [56]. The best known example is Wheeler-Feynman electrodynamics, where the constraint that only solutions that are Taylor expandable in powers of v/c are permitted, corresponds to the limit of infinite propagation speed of instantaneous interaction in the original theory [59].

In the realm of effective theories, Lagrangians with higher-order derivatives have also been proposed and profoundly studied in order to parameterize possible deviations from the standard model. It has been proved that up to the energy scale at the effective Lagrangian level, all the disturbing problems associated with higher-derivative theories are absent [60]. This is indeed the case demonstrated in the previous chapters. However, how the spectra of deformed nuclei will be corrected at higher order by higher-order derivatives in the effective theory still arouses our interest.

In order to see how higher derivatives bring about new effects in our effective theory of deformed nuclei, one basically has two directions to go. First, a formalism that deals with higher-derivative Lagrangian can be utilized to derive the Hamiltonian. A formalism for this task – Ostrogradsky formalism – has been long established [61]. Essentially, it treats all the time derivatives as independent coordinates and performs a field transformation (namely, a canonical transformation) on them. This treatment brings in additional degrees of freedom. The second way is more straightforward, which is to apply the equations of motion to substitute the higher-order derivative terms and reduce the order of derivatives of the Lagrangian. Physically, these two approaches are expected to be equivalent, which has been indicated in Ref. [60]. In this chapter we closely follow these two approaches.

The structure of this chapter is arranged as follows. First, the transformation properties of terms with second-order derivatives are proved to satisfy the requirement of constructing rotationally invariant Lagrangians. Then, a general theory that handles higher-derivative Lagrangians is employed to eliminate the second-derivative terms in our effective Lagrangian. This, however, leads to an unsatisfactory result for

deformed nuclei, and is subject to further discussion. Furthermore, a new variable transformation that maintains the time invariance of even-even nuclei is found. It can be proved to be correct up to NLO in an *adhoc* way. Nevertheless, this transformation, while derived purely mathematically, still needs meaningful physical interpretation.

A.1 Transformation properties

The transformation properties of the "building blocks" of a rotationally invariant Lagrangian with only first-order derivatives have already been derived [13]. Here we give the transformation properties of new "building blocks" that contains second-order time derivatives.

First, let us look at second-order time derivatives of Nambu-Goldstone modes, e.g. D_tE_+ and D_tE_-. The transformation rule of the covariant operator D_t is

$$\widetilde{D}_t = hD_th^{-1} \ , \tag{A.1}$$

where $h = \exp(-i\gamma\hat{J}_z)$ is the arbitrary rotation operator around the symmetry axis of the deformed nucleus, parameterized nonlinearly by the Euler angles. Thus, the transformation properties of $D_tE_\pm$ are

$$\begin{aligned} \widetilde{D}_t\widetilde{E}_+ = &\quad e^{-i\gamma}D_te^{i\gamma}e^{-i\gamma}E_+ &&= e^{-i\gamma}D_tE_+ \ , \\ \widetilde{D}_t\widetilde{E}_- = &\quad e^{i\gamma}D_te^{-i\gamma}e^{i\gamma}E_- &&= e^{i\gamma}D_tE_- \ . \end{aligned} \tag{A.2}$$

Indeed, they have the desired transformation under a rotation around the symmetry axis as expected. In addition to Nambu-Goldstone modes, the second-order covariant derivative of quadrupole field $D_t^2\phi$ is also well transformed, as shown below

$$\widetilde{D}_t^2\widetilde{\phi} = hD_th^{-1}hD_th^{-1}h\phi = hD_t^2\phi \ . \tag{A.3}$$

These transformation properties justify our motivation to include second-order derivative terms into the effective Lagrangian of deformed nuclei. Surely, the transformation rules can be easily generalized to terms with even higher-order(>2) time derivatives.

A.2 Second-order derivatives in Nambu-Goldstone modes

In this chapter, we investigate the higher-derivative Lagrangians that involve $D_t E_\pm$, i.e. Nambu-Goldstone modes. Terms containing $D_t^2\phi$ regarding the quadrupole vibrational degrees of freedom are more complicated and not discussed here.

For odd nuclei, terms entering the effective Lagrangian are not time-reversal invariant. They are $E_+ D_t E_-$ and $E_- D_t E_+$

$$\begin{aligned} E_+ D_t E_- &= E_+ \partial_t E_- + i E_z E_+ E_- \,, \\ E_- D_t E_+ &= E_- \partial_t E_+ - i E_z E_+ E_- \,. \end{aligned} \tag{A.4}$$

Both of them are complex. Only the linear combinations of them yield purely real or imaginary values

$$E_+ D_t E_- + E_- D_t E_+ = E_+ \partial_t E_- + E_- \partial_t E_+ = \partial_t (E_+ E_-) \,, \tag{A.5}$$

$$\begin{aligned} E_+ D_t E_- - E_- D_t E_+ &= E_+ \partial_t E_- - E_- \partial_t E_+ + 2 i E_z E_+ E_- \\ &= 2i(\ddot{\alpha}\dot{\beta}\sin\beta + \dot{\alpha}\dot{\beta}^2\cos\beta - \dot{\alpha}\ddot{\beta}\sin\beta) \\ &\quad -2i\dot{\alpha}\cos\beta(\dot{\alpha}^2\sin^2\beta + \dot{\beta}^2) \\ &= 2i(\ddot{\alpha}\dot{\beta}\sin\beta - \dot{\alpha}\ddot{\beta}\sin\beta - \dot{\alpha}^3\sin^2\beta\cos\beta) \,. \end{aligned} \tag{A.6}$$

Note that the sum of $E_+ D_t E_-$ and $E_- D_t E_+$ is just a total time derivative that's neglectable from the Lagrangian. Of interest is the subtraction of them, which should

be included by the Lagrangian with a parameter scaling it.

$$\begin{aligned} L^{(oo)} &= \frac{C_0}{2}(\dot{\alpha}^2\sin^2\beta+\dot{\beta}^2) - q\dot{\alpha}\cos\beta \\ &\quad +\frac{iC_1}{2}(E_+D_tE_- - E_-D_tE_+) \\ &= \frac{C_0}{2}(\dot{\alpha}^2\sin^2\beta+\dot{\beta}^2) - q\dot{\alpha}\cos\beta \\ &\quad +C_1(\dot{\alpha}^3\sin^2\beta\cos\beta+\dot{\alpha}\ddot{\beta}\sin\beta-\ddot{\alpha}\dot{\beta}\sin\beta)\ . \end{aligned} \tag{A.7}$$

The corresponding power counting is

$$\begin{aligned} E_+D_tE_- \sim E_-D_tE_+ &\sim \xi^3\ , \\ C_1 &\sim \Omega^{-2}\ , \end{aligned} \tag{A.8}$$

so that the second-order derivative term scale as

$$C_1(E_+D_tE_- - E_-D_tE_+) \sim \xi\left(\frac{\xi}{\Omega}\right)^2\ . \tag{A.9}$$

As the simplest Lagrangian containing 2nd-order derivatives has been written out for odd nuclei, we stop here temporarily and turn to the even-even nuclei, which is invariant under time reversal. They require the total order of time derivatives to be even. The only term meeting this requirement is $(D_tE_+)(D_tE_-)$,

$$(D_tE_+)(D_tE_-) = \ddot{\alpha}^2\sin^2\beta+\ddot{\beta}^2+2\dot{\alpha}^2\ddot{\beta}\sin\beta\cos\beta+\dot{\alpha}^4\sin^2\beta\cos^2\beta\ . \tag{A.10}$$

The Lagrangian for even-even nuclei is

$$\begin{aligned} L^{(ee)} &= \frac{C_0}{2}(\dot{\alpha}^2\sin^2\beta+\dot{\beta}^2)+\frac{C_2}{4}(\dot{\alpha}^2\sin^2\beta+\dot{\beta}^2)^2 \\ &\quad +C_3(D_tE_+)(D_tE_-)\ . \end{aligned} \tag{A.11}$$

Still, we have the following power counting

$$\begin{aligned}(D_tE_+)(D_tE_-) &\sim \xi^4\,,\\ C_3 &\sim \xi^{-1}\Omega^{-2}\,,\\ C_3(D_tE_+)(D_tE_-) &\sim \xi\left(\frac{\xi}{\Omega}\right)^2\,.\end{aligned} \tag{A.12}$$

As we can see, the new second-order derivative term scales equivalently as NLO of Nambu-Goldstone modes, which is a factor of two powers of ξ/Ω smaller than the LO Nambu-Goldstone modes.

Now the Lagrangians of both even-even and odd nuclei have been obtained, which contain time derivatives of Nambu-Goldstone modes up to 2nd order. One naturally quests about what effects do the 2nd-derivative terms bring to the spectrum of deformed nuclei. This cannot be answered until we obtain the Hamiltonian and calculate the eigenenergy. A general method that aims to reduce the higher-order derivatives in Lagrangian has been employed to convert Eq.A.7 and Eq.A.11 to an ordinary one with only 1st-order time derivatives. It uses the equations of motion and substitutes the higher-order derivatives in the Lagrangian with lower-order derivatives. This is also considered as equivalent to redefining the position variables (i.e. making a variable transformation) that eliminates the higher-order derivatives. Readers are referred to Ref. [62] for a detailed illustration of the method.

However, it turns out that the ordinary Lagrangian derived by the method of position variables redefinition suffers some problems that violate the physical properties of deformed nuclei. For instance, in the case of even-even nuclei we have the following substitution of the accelerations based on the method in [62]

$$\begin{aligned}\ddot{\alpha} &\to -2\dot{\alpha}\dot{\beta}\cot\beta\,,\\ \ddot{\beta} &\to \dot{\alpha}^2\sin\beta\cos\beta\,.\end{aligned} \tag{A.13}$$

This is followed by a Lagrangian based on Eq.A.11 which has a very concise form

$$\begin{aligned} L_{tr}^{(ee)} &= \frac{C_0}{2}(\dot{\alpha}^2\sin^2\beta+\dot{\beta}^2)+\frac{C_2}{4}(\dot{\alpha}^2\sin^2\beta+\dot{\beta}^2)^2 \\ &\quad +4C_3\dot{\alpha}^2\cos^2\beta(\dot{\alpha}^2\sin^2\beta+\dot{\beta}^2)\ , \end{aligned} \tag{A.14}$$

i.e. the second-order derivative term can be written as simply as $E_z^2E_+E_-$, represented in terms of Nambu-Goldstone fields. However, this Lagrangian is not rotationally invariant. Therefore, the validity of the transformation needs to be further examined.

Whereas the transformation of acceleration Eq.A.13 still preserves the time-reversal invariance in even-even nuclei, a transformation that breaks the time invariance has also been derived for odd nuclei. The results are under further study. In the next section, we present a variable transformation that differs from Eq.A.13 for even-even nuclei, but also keeps time-reversal invariance. The derivation of the transformation is independent of any formal theory, but can be proved to be valid in terms of reducing the higher-order derivatives in our effective Lagrangian. Correction terms of higher order than NLO Nambu-Goldstone modes are always neglected in the procedure.

A.3 Variable transformation

In this section we give a variable transformation that converts the effective Lagrangian Eq.A.11 of even-even nuclei to a single time-derivative one. The transformation does not rely on any formal theories that have been developed in literatures now. The entire transformation can be decomposed into two steps, which gradually achieve our purpose to reduce the derivative order.

We start from the Lagrangian

$$\begin{aligned} L^{(ee)} =& \frac{C_0}{2}(\dot{\alpha}^2\sin^2\beta+\dot{\beta}^2)+\frac{C_2}{4}(\dot{\alpha}^2\sin^2\beta+\dot{\beta}^2)^2 \\ &+C_3(\ddot{\alpha}^2\sin^2\beta+\ddot{\beta}^2+2\dot{\alpha}^2\ddot{\beta}\sin\beta\cos\beta+\dot{\alpha}^4\sin^2\beta\cos^2\beta)\ . \end{aligned} \tag{A.11}$$

If we make the following transformation

$$\begin{aligned} \alpha &\rightarrow \alpha' + \zeta\ddot{\alpha}' \,, \\ \beta &\rightarrow \beta' + \eta\ddot{\beta}' \,, \end{aligned} \tag{A.15}$$

with ζ and η being unknown parameters to be determined. Later it will be more clear why we adopt this form of transformation. The LO Lagrangian after the transformation becomes

$$\begin{aligned} L_{\rm LO}^{(ee)} &= \frac{C_0}{2}(\dot{\alpha}^2 \sin^\beta + \dot{\beta}^2) \\ &= \frac{C_0}{2}\big((\dot{\alpha}'^2 + 2\zeta\dot{\alpha}'\dddot{\alpha}')(\sin^2\beta' + 2\eta\ddot{\beta}'\cos\beta'\sin\beta') + (\dot{\beta}'^2 + 2\eta\dot{\beta}'\dddot{\beta}')\big) \\ &= \frac{C_0}{2}(\dot{\alpha}'^2\sin^2\beta' + \dot{\beta}'^2) + C_0\zeta\dot{\alpha}'\dddot{\alpha}'\sin^2\beta' \\ &\quad + C_0\eta\dot{\alpha}'^2\ddot{\beta}'\cos\beta'\sin\beta' + C_0\eta\dot{\beta}'\dddot{\beta}' \,. \end{aligned} \tag{A.16}$$

Note that the Lagrangian is truncated at the order $\xi\left(\frac{\xi}{\Omega}\right)^2$, which is NLO of Nambu-Goldstone modes. The power counting of the parameters ζ and η must be enforced as

$$\zeta \sim \eta \sim \Omega^{-2} \,. \tag{A.17}$$

All the terms that are of higher order in the Lagrangian have been neglected. For the Lagrangian terms at NLO, the expression does not change except that α and β are simply replaced by α' and β'. This is because that $\zeta\ddot{\alpha}'$ and $\eta\ddot{\beta}'$ are already higher-order corrections to α' and β'.

There exist third-order derivatives of Nambu-Goldstone modes (scaling as $\dddot{\alpha} \sim \dddot{\beta} \sim \xi^3$), which seem not easy to handle. However, recall that the Lagrangian is to be integrated as action to apply the minimum action principle. This means performing a partial integration of any single term in the Lagrangian will not change the equations of motion or final results. More specifically, the products of first-order derivative

and third-order derivative can be integrated and converted to quadratics of second-order derivative. Let's take a detailed look at the two terms that contain 3rd-order derivatives in Eq.A.16

$$\int_i^f C_0\zeta\dot{\alpha}'\dddot{\alpha}'\sin^2\beta' = C_0\zeta\dot{\alpha}'\ddot{\alpha}'\sin^2\beta'\Big|_i^f - \int_i^f C_0\zeta'\ddot{\alpha}'\partial_t(\dot{\alpha}\sin^2\beta')\,, \quad \text{(A.18)}$$

$$\int_i^f C_0\eta\dot{\beta}'\dddot{\beta}' = C_0\eta\dot{\beta}'\ddot{\beta}'\Big|_i^f - \int_i^f C_0\eta'\ddot{\beta}'\partial_t\dot{\beta}'\,. \quad \text{(A.19)}$$

The first terms in both Eq.A.18 and Eq.A.19 are zero according to the variational principle. Hence, the third-derivative terms in the Lagrangian can be replaced with

$$C_0\zeta\dot{\alpha}'\dddot{\alpha}'\sin^2\beta' \rightarrow -C_0\zeta'\ddot{\alpha}'^2\sin^2\beta' - 2C_0\zeta'\ddot{\alpha}'\dot{\alpha}'\beta'\sin\beta'\cos\beta'\,, \quad \text{(A.20)}$$

$$C_0\eta\dot{\beta}'\dddot{\beta}' \rightarrow -C_0\eta'\ddot{\beta}'^2\,. \quad \text{(A.21)}$$

The entire Lagrangian becomes

$$\begin{aligned} L_{\rm NLO}^{(ee)} &= L_{\rm LO}^{(ee)} + \frac{C_2}{4}(\dot{\alpha}'^2\sin^2\beta' + \dot{\beta}'^2)^2 \\ &\quad + C_3(\ddot{\alpha}'^2\sin^2\beta' + \ddot{\beta}'^2 + 2\dot{\alpha}'^2\ddot{\beta}'\sin\beta'\cos\beta' + \dot{\alpha}'^4\sin^2\beta'\cos^2\beta')\,, \quad \text{(A.22)} \\ L_{\rm LO}^{(ee)} &= \frac{C_0}{2}(\dot{\alpha}'^2\sin^2\beta' + \dot{\beta}'^2) \\ &\quad - C_0\zeta\ddot{\alpha}'^2\sin^2\beta' - 2C_0\zeta\ddot{\alpha}'\dot{\alpha}'\beta'\sin\beta'\cos\beta' \\ &\quad + C_0\eta\dot{\alpha}'^2\ddot{\beta}'\cos\beta'\sin\beta' - C_0\eta\ddot{\beta}'^2\,. \quad \text{(A.23)} \end{aligned}$$

Note that the transformed Lagrangian Eq.A.23 from the LO Lagrangian is already not purely of LO. It contains higher-order correction terms originating from the transformation. If we let $\zeta = \eta = C_3/C_0$, we are immediately able to cancel several terms in Eq.A.22 and get

$$
\begin{aligned}
L_{\rm NLO}^{(ee)} &= \frac{C_0}{2}(\dot\alpha'^2\sin^2\beta'+\dot\beta'^2)+\frac{C_2}{4}(\dot\alpha'^2\sin^2\beta'+\dot\beta'^2)^2 \\
&\quad +3C_3\dot\alpha'^2\ddot\beta'\sin\beta'\cos\beta'+C_3\dot\alpha'^4\sin^2\beta'\cos^2\beta' \\
&\quad -2C_3\ddot\alpha'\dot\alpha'\beta'\sin\beta'\cos\beta' \ . \qquad \text{(A.24)}
\end{aligned}
$$

This Lagrangian contains less 2nd-order derivative terms compared to the original one Eq.A.11. We will soon see as the second step another single variable transformation quickly eliminates all the 2nd-order derivatives at the NLO in Nambu-Goldstone modes.

We have already successfully implemented the first transformation

$$
\begin{aligned}
\alpha &\rightarrow \alpha'+\frac{C_3}{C_0}\ddot\alpha' \ , \\
\beta &\rightarrow \beta'+\frac{C_3}{C_0}\ddot\beta' \ . \qquad \text{(A.25)}
\end{aligned}
$$

The second transformation is

$$
\begin{aligned}
\alpha' &\rightarrow \alpha'' \ , \\
\beta' &\rightarrow \beta''+\varepsilon\dot\alpha''^2\sin\beta''\cos\beta'' \ . \qquad \text{(A.26)}
\end{aligned}
$$

α' keeps the same, but β' transforms in a slightly more complicated way. We explicitly write out the transformation rule of $\dot\beta'$ derived from Eq.A.26

$$
\dot\beta' \rightarrow \dot\beta''+2\varepsilon\dot\alpha''\ddot\alpha''\sin\beta''\cos\beta''+\varepsilon\dot\alpha''^2\dot\beta''\cos 2\beta'' \ . \qquad \text{(A.27)}
$$

Before we put it into Eq.A.24, a little more manipulation on it needs to be done. Again, we conduct a partial integration on one of its term

$$
\int_i^f \dot\alpha'^2\ddot\beta'\sin\beta'\cos\beta' = \dot\alpha'^2\dot\beta'\sin\beta'\cos\beta'\Big|_i^f-\int_i^f\dot\beta'\partial_t(\dot\alpha'^2\sin\beta'\cos\beta') \ , \qquad \text{(A.28)}
$$

which yields

$$\dot{\alpha}'^2\ddot{\beta}'\sin\beta'\cos\beta' \to -2\dot{\alpha}'\ddot{\alpha}'\dot{\beta}'\sin\beta'\cos\beta' - \dot{\alpha}'^2\dot{\beta}'^2\cos 2\beta' \ . \tag{A.29}$$

The Lagrangian Eq.A.24 turns into

$$\begin{aligned} L_{\rm NLO}^{(ee)} &= \frac{C_0}{2}(\dot{\alpha}'^2\sin^2\beta' + \dot{\beta}'^2) + \frac{C_2}{4}(\dot{\alpha}'^2\sin^2\beta' + \dot{\beta}'^2)^2 \\ &\quad -8C_3\dot{\alpha}'\ddot{\alpha}'\dot{\beta}'\sin\beta'\cos\beta' - 3C_3\dot{\alpha}'^2\dot{\beta}'^2\cos 2\beta' \\ &\quad +C_3\dot{\alpha}'^4\sin^2\beta'\cos^2\beta' \ . \end{aligned} \tag{A.30}$$

Substitute Eq.A.29 into the above Lagrangian

$$\begin{aligned} L_{\rm NLO}^{(ee)} &= \frac{C_0}{2}(\dot{\alpha}''^2\sin^2\beta'' + \dot{\beta}''^2) + \frac{C_2}{4}(\dot{\alpha}''^2\sin^2\beta'' + \dot{\beta}''^2)^2 \\ &\quad -8C_3\dot{\alpha}''\ddot{\alpha}''\dot{\beta}''\sin\beta''\cos\beta'' - 3C_3\dot{\alpha}''^2\dot{\beta}''^2\cos 2\beta'' \\ &\quad +C_3\dot{\alpha}''^4\sin^2\beta''\cos^2\beta'' + C_0\varepsilon\dot{\alpha}''^4\sin^2\beta''\cos^2\beta'' \\ &\quad +2C_0\varepsilon\dot{\alpha}''\ddot{\alpha}''\dot{\beta}''\sin\beta''\cos\beta'' + C_0\varepsilon\dot{\alpha}''^2\dot{\beta}''^2\cos 2\beta'' \ . \end{aligned} \tag{A.31}$$

Clearly, the 2nd-order derivative terms can be eliminated by letting $\varepsilon = 4C_3/C_0$. The final Lagrangian is

$$\begin{aligned} L_{\rm NLO}^{(ee)} &= \frac{C_0}{2}(\dot{\alpha}''^2\sin^2\beta'' + \dot{\beta}''^2) + \frac{C_2}{4}(\dot{\alpha}''^2\sin^2\beta'' + \dot{\beta}''^2)^2 \\ &\quad +C_3\dot{\alpha}''^2\dot{\beta}''^2\cos 2\beta'' + 5C_3\dot{\alpha}''^4\sin^2\beta''\cos^2\beta'' \ . \end{aligned} \tag{A.32}$$

For cleanness we omit the primes

$$\begin{aligned} L_{\rm NLO}^{(ee)} &= \frac{C_0}{2}(\dot{\alpha}^2\sin^2\beta + \dot{\beta}^2) + \frac{C_2}{4}(\dot{\alpha}^2\sin^2\beta + \dot{\beta}^2)^2 \\ &\quad +C_3\dot{\alpha}^2\dot{\beta}^2\cos 2\beta + 5C_3\dot{\alpha}^4\sin^2\beta\cos^2\beta \ . \end{aligned} \tag{A.33}$$

The transformations Eq.A.25 and Eq.A.26 can be combined into a single transformation shown as follows

$$\begin{aligned} \alpha &\rightarrow \alpha'' + \frac{C_3}{C_0}\ddot{\alpha}'' \,, \\ \beta &\rightarrow \beta'' + \frac{C_3}{C_0}\ddot{\beta}'' + \frac{4C_3}{C_0}\dot{\alpha}''^2 \sin\beta'' \cos\beta'' \,. \end{aligned} \tag{A.34}$$

Again, as what has been done in deriving the Lagrangian, we also only keep terms of up to NLO Nambu-Goldstone modes, i.e. the factor ξ^2/Ω^2 smaller than the leading-order variables.

Many questions can be naturally raised about the transformed Lagrangian Eq.A.33. Whether this Lagrangian is rotationally invariant for the axially deformed nuclei? What is the conserved quantities for this Lagrangian? What is the expression of angular momentum, and is it still conserved as derived in [13]? What is the Hamiltonian? What does the rotational spectrum look like for this Lagrangian? How is it altered by the second-order time derivatives compared to the results of first-derivative theory? Furthermore, one is always supposed to make necessary physical interpretation when performing a variable transformation. These are questions to be answered in future study. We will give a thread of thought toward the solutions in the next section.

A.4 Conserved quantity

In previous chapters, we have shown that the energy spectrum of deformed nuclei is closely connected with the conserved quantity, namely the angular momentum. Whether it is for even-even or odd nuclei, there is $l(l+1)$ characterizing the rotational band, as in Eq.3.12 and Eq.4.22. $l(l+1)$ is either exactly the square of total angular momentum or differs with it by only one term [13]. Thus, in our effective theory of deformed nuclei, the angular momentum plays a central role in helping us understand the physical properties and energy spectrum more precisely.

For the Lagrangian for even-even nuclei with new 2nd-order derivative entered, we have obtained a variable transformation that preserves the time-reversal invariance and eliminates the 2nd-order derivatives in the original Lagrangian. The final Lagrangian Eq.A.33 contains only 1st-order derivatives. Before one starts to compute the Hamiltonian and eigenenergies, it may be helpful to ask what the corresponding conserved quantities are. It is expected that the conserved quantity is still total angular momentum, and the energy spectrum is a simple formula that can be represented by the angular momentum. There are two paths that we can follow to answer this question.

First, Eq.A.33 gives a normal Lagrangian with only 1st-order derivatives. The expression of conserved quantity has been derived [13] by applying Noether's theorem,

$$Q_k = \sum_{i=1}^{N} \frac{\partial L}{\partial \dot{q}_\nu} \hat{M}_{\nu k} = \sum_{i=1}^{N} p_\nu \hat{M}_{\nu k} \ , \tag{A.35}$$

where the matrix $\hat{M}_{\nu k}$ equals

$$\hat{M} = \begin{pmatrix} -\cot\beta\cos\alpha & -\cot\beta\sin\alpha & 1 \\ -\sin\alpha & \cos\alpha & 0 \end{pmatrix} . \tag{A.36}$$

This matrix transforms the infinitesimal rotation around Cartesian axes $\delta\omega_{x,y,z}$ to the infinitesimal change in terms of the Euler angles $\delta\alpha$ and $\delta\beta$, parameterized also by Euler angles. The conjugate momenta p_ν in Eq.A.35 can be calculated straightforwardly based on Eq.A.33

$$\begin{aligned} q_\alpha &= C_0\dot{\alpha}\sin^2\beta + C_2\dot{\alpha}\sin^2\beta(\dot{\alpha}^2\sin^2\beta + \dot{\beta}^2) \\ &\quad +2C_3\dot{\alpha}\dot{\beta}^2\cos 2\beta + 20C_3\dot{\alpha}^3\sin^2\beta\cos^2\beta \ , \\ q_\beta &= C_0\beta + C_2\dot{\beta}(\dot{\alpha}^2\sin^2\beta + \dot{\beta}^2) + 2C_3\dot{\alpha}^2\dot{\beta}\cos 2\beta \ . \end{aligned} \tag{A.37}$$

Then, we can apply the inversion of the transformation Eq.A.34

$$\begin{aligned} \alpha &\rightarrow \alpha'' - \frac{C_3}{C_0}\ddot{\alpha}'' \,, \\ \beta &\rightarrow \beta'' - \frac{C_3}{C_0}\ddot{\beta}'' - \frac{4C_3}{C_0}\dot{\alpha}''^2 \sin\beta'' \cos\beta'' \,, \end{aligned} \tag{A.38}$$

and immediately obtain the expression of angular momentum Q_k in terms of the original Euler angles α and β.

This is the first approach to calculate the angular momentum, which primarily relies on the variable transformation. Everything in the original theory can be simply substituted through applying the variable transformation, which leads us to the results for the 2nd-derivative Lagrangian. The second approach to derive the conserved quantity is more straightforward. We can directly begin with the original Lagrangian Eq.A.11, and calculate its conserved quantity without using the variable transformation. Like the 1st-derivative Lagrangian, Noether's theorem can be directly applied to the 2nd-derivative Lagrangian.

Here, we will not calculate the conserved quantity (i.e. angular momentum) in detail for the original 2nd-derivative Lagrangian. Rather, we only present the way to apply Noether's theorem that derives the conserved quantity for 2nd-derivative Lagrangian, i.e. a general expression for the conserved quantity. Calculations for specific case should be quite straightforward with our final result. For a Lagrangian with 2nd-order derivatives of position, a variation of it according to the chain rule can be expressed as

$$\delta L = \frac{\partial L}{\partial q_k}\delta q_k + \frac{\partial L}{\partial \dot{q}_k}\delta \dot{q}_k + \frac{\partial L}{\partial \ddot{q}_k}\delta \ddot{q}_k \,, \tag{A.39}$$

In addition, there is the equation of motion

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}_k} = 0 \,. \tag{A.40}$$

Putting it into Eq.A.39 we get

$$\delta L = \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}_k} \right) \delta q_k + \frac{\partial L}{\partial \dot{q}_k}\delta \dot{q}_k + \frac{\partial L}{\partial \ddot{q}_k}\delta \ddot{q}_k \ . \tag{A.41}$$

Note that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\delta q_k \right) = \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} \right) \delta q_k + \frac{\partial L}{\partial \dot{q}_k}\delta \dot{q}_k \ , \tag{A.42}$$

and

$$\frac{\partial L}{\partial \ddot{q}_k}\delta \ddot{q}_k = \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}_k}\delta \dot{q}_k \right) - \left(\frac{d}{dt}\frac{\partial L}{\partial \ddot{q}_k} \right) \delta \dot{q}_k \ . \tag{A.43}$$

The variation of Lagrangian can be written as

$$\delta L = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\delta q_k \right) + \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}_k}\delta \dot{q}_k \right) - \frac{d}{dt}\left(\left(\frac{d}{dt}\frac{\partial L}{\partial \ddot{q}_k} \right) \delta q_k \right) \ . \tag{A.44}$$

Applying Noether's theorem we get

$$0 = \delta L = \frac{d\delta Q_k}{dt} \ , \tag{A.45}$$

where δQ_k is the variation of the component of angular momentum and equals

$$\delta Q_k = \left(\frac{\partial L}{\partial \dot{q}_k} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}_k} \right) \delta q_k + \frac{\partial L}{\partial \ddot{q}_k}\delta \dot{q}_k \ . \tag{A.46}$$

Now that δq_k has been given [13]

$$\delta q_k = M_{k\nu}\delta \omega_\nu \ . \tag{A.47}$$

It should be pointed out that $\delta \dot{\omega}_\nu = 0$, because the Cartesian coordinates do not change with time. Therefore,

$$\delta \dot{q}_k = \dot{M}_{k\nu}\delta \omega_\nu \ . \tag{A.48}$$

Finally, the component of angular momentum Q_ν is

$$Q_\nu = \left(\frac{\partial L}{\partial \dot{q}_k} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{q}_k} \right) M_{k\nu} + \frac{\partial L}{\partial \ddot{q}_k} \dot{M}_{k\nu} \ , \tag{A.49}$$

where $M_{k\nu}$ is given by Eq.A.36 and $\dot{M}_{k\nu}$ is the derivative of it with respect to time.

Till now, we have developed the formalism to derive the conserved quantity angular momentum for our 2nd-derivative effective Lagrangian of deformed nuclei, on the basis of Noether's theorem. It is straightforward to apply Eq.A.49 directly to the Lagrangians Eq.A.7 and Eq.A.11 to compute the conserved angular momentum for both even-even and odd nuclei. The results can also be compared with that obtained by the first method, which utilizes the variable transformation. The consistency between them will verify the correctness of both approaches, and give rise to the right expression of the conserved quantity.

Vita

Jialin Zhang was born and raised in Shanghai, People's Republic of China, to the parents of Jiguo Zhang and Guifang Kong. After graduating from the High School Attached to Shanghai JiaoTong University in 2005, he enrolled in the Department of Physics at Shanghai JiaoTong University, China. Four years later he received his Bachelor of Science degree in Optics Information and Technology at SJTU. Upon finishing the undergraduate study at SJTU, he headed to the United States and began the graduate study at the University of Tennessee, Knoxville. For the first year at UTK, he worked as a research assistant in the Advanced Laser Diagnostics Laboratory under the supervision of Dr. Zhili Zhang at the Department of Mechanical, Aerospace and Biomedical Engineering. In the summer of 2010, he transferred to the Department of Physics and Astronomy at UTK and continued to pursue a graduate degree in theoretical physics. While serving as a graduate teaching assistant there, he also obtained rich opportunities in both touching the forefronts of theoretical and computational researches and expanding his knowledge in diverse fields. He joined with Dr. Thomas Papenbrock in the fall of 2011 to study the effective theory on deformed nuclei. He will be graduating with the Master of Science degree in nuclear physics from UTK in July 2012, and is looking forward to a career in information technology industry back in China.